

On the Secrecy Capacity of a MIMO Gaussian Wiretap Channel with a Cooperative Jammer

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Abstract—We study the secrecy capacity of a helper-assisted Gaussian wiretap channel with a source, a legitimate receiver, an eavesdropper and an external helper, where each terminal is equipped with multiple antennas. Determining the secrecy capacity in this scenario generally requires solving a nonconvex secrecy rate maximization (SRM) problem. To deal with this issue, we first reformulate the original SRM problem into a sequence of convex subproblems. For the special case of single-antenna legitimate receiver, we obtain the secrecy capacity via a combination of convex optimization and one-dimensional search, while for the general case of multi-antenna legitimate receiver, we propose an iterative solution. To gain more insight into how the secrecy capacity of a helper-assisted Gaussian wiretap channel behaves, we examine the achievable secure degrees of freedom (s.d.o.f.) and obtain the maximal achievable s.d.o.f. in closed-form. We also derive a closed-form solution to the original SRM problem which achieves the maximal s.d.o.f.. Numerical results are presented to illustrate the efficacy of the proposed schemes.

I. INTRODUCTION

The area of physical (PHY) layer security has been pioneered by Wyner [1], who introduced the wiretap channel and quantified security with the maximal achievable secrecy rate (also known as the secrecy capacity) at which the legitimate receiver can correctly decode the source message, while the eavesdropper can retrieve almost no information. Results in [2] further show that for the classical source-destination-eavesdropper Gaussian wiretap channel, the secrecy capacity is zero when the quality of the legitimate channel is worse than that of the eavesdropper's channel. One way to achieve non-zero secrecy rate in the latter case is to introduce external helpers which act as cooperative jammers [3]. By transmitting jamming signals the external helpers degrade the eavesdropper's channel without hurting the legitimate channel, thus allowing secret communication even when the eavesdropper's channel has a much better quality. Works along these lines include [4]–[7] which consider one external helper and [8]–[14] which consider the case of multiple external helpers. More complex relaying scenarios are considered in [15]–[18] where the jamming signal is sent in the relaying phase, or in both the broadcasting phase and the relaying phase.

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Although cooperative jamming approaches improve the secrecy rate, their advantage comes from optimally designed input covariance matrices, which are difficult to obtain due to the nonlinear fractional nature of the problem. To address this issue, for the single-antenna eavesdropper case, [10]–[12] propose a suboptimal but cost efficient null-space jamming scheme that spreads the jamming signal within the null-space of the legitimate receiver's channel, while [6]–[9], [18] design algorithms to get the optimal solution using a combination of convex optimization and one-dimensional search. For the multi-antenna eavesdropper case, [13], [14] design the jamming signals so that they align into a pre-specified jamming subspace at the legitimate receiver, while spanning the whole received signal space at the eavesdropper. This approach allows the legitimate receiver to completely remove the interference by projecting the received signal to the secrecy subspace, while confounding the eavesdropper. Still for the multi-antenna eavesdropper case, the work of [4] provides a closed-form expression for the structure of the jamming signal covariance matrix that guarantees secrecy rate larger than the secrecy capacity of the wiretap channel with no jamming signals. The results of [4] are obtained under the power covariance constraint.

In this paper, we consider a multi-input multi-output (MIMO) Gaussian wiretap channel with one external multi-antenna helper as in [4]. Different from [4], we investigate the secrecy rate maximization (SRM) problem under an average power constraint. To the best of the authors' knowledge, determining the exact secrecy capacity of a helper-assisted MIMO Gaussian wiretap channel has not been previously addressed. We first address the problem for the special case of single-antenna legitimate receiver. By decomposing the original nonconvex SRM problem into a sequence of convex subproblems, we obtain the optimal solution to the original SRM problem via a combination of convex optimization and one-dimensional search. For the general case of multi-antenna legitimate receiver, we propose an iterative algorithm to solve the original SRM problem via employing the Gauss-Seidel approach, which successively optimizes each variable while the other variables are kept fixed. Specifically, each subproblem is convex and admits an optimal solution. Though the proposed iterative algorithm provides no guarantee of finding the global optimal solution, it constitutes an efficient way for attaining a meaningful achievable secrecy rate.

In order to gain more insight into how the secrecy capacity of a helper-assisted MIMO Gaussian wiretap channel behaves, we examine the rate at which the secrecy capacity scales with $\log(P)$, i.e., the maximal achievable secure degrees of freedom

(s.d.o.f.) [19]. To this end, we first introduce an alternative optimization problem, i.e., maximizing the dimension of the subspace spanned by the message signal received at the legitimate receiver, under the constraints that the message and jamming signals lie in different subspaces at the legitimate receiver, but are aligned into the same subspace at the eavesdropper. We then give a critical lemma, proving that the maximal achievable objective value of the newly introduced optimization problem equals the maximal achievable s.d.o.f.. Consequently, the original s.d.o.f. maximization reduces to the newly introduced optimization problem. Subsequently, we solve analytically the newly introduced optimization problem, thus obtaining the maximal achievable s.d.o.f. of the helper-assisted MIMO Gaussian wiretap channel in closed-form. Further, we derive an analytical solution to the original SRM problem, which achieves the maximal s.d.o.f.. Our analytical results prove that for the special case of single-antenna legitimate receiver, a s.d.o.f. of 1 can be achieved if and only if $N_e < N_a + N_j - 1$; for the case of multi-antenna legitimate receiver, the maximal achievable s.d.o.f. is zero if and only if $N_e \geq N_a + N_j$.

We should note that the s.d.o.f. for the helper-assisted Gaussian wiretap channel has also been investigated in [14], [20]–[22]. Different from our work, the work of [14] studies a scenario in which a large number of helpers is available, and exploits multiuser diversity via opportunistic helper selection to enhance security. The works of [20], [21] consider a scenario in which each terminal is equipped with one antenna, while the work of [22] considers the special scenario in which the source, the legitimate receiver and the eavesdropper are equipped with the same number of antennas. Further, the works of [20]–[22] examine the s.d.o.f. based on real interference alignment, while our work is based on spatial interference alignment.

The rest of this paper is organized as follows. In Section II, we describe the system model for the MIMO Gaussian wiretap channel with one external multi-antenna helper, and formulate the secrecy rate maximization problem. In Section III, we consider the special case of single-antenna legitimate receiver. We investigate the secrecy rate maximization problem, and examine the conditions under which a secure degrees of freedom equal to 1 can be achieved. In Section IV, we consider the general case of multi-antenna legitimate receiver, investigate the secrecy rate maximization problem, and examine the maximal achievable secure degrees of freedom. Numerical results are provided in Section V and conclusions are drawn in Section VI.

Notation: \mathbf{A}^H , $\text{tr}\{\mathbf{A}\}$ and $\text{rank}\{\mathbf{A}\}$ stand for the hermitian transpose, trace and rank of the matrix \mathbf{A} , respectively; $\mathbf{A}(:, j)$ indicates the j -th column of \mathbf{A} while $\mathbf{A}(:, i : j)$ denotes the columns from i to j of \mathbf{A} . $\text{span}(\mathbf{A})$ and $\text{span}(\mathbf{A})^\perp$ are the subspace spanned by the columns of \mathbf{A} and its orthogonal complement, respectively; $\text{null}(\mathbf{A})$ denotes the null space of \mathbf{A} ; $\text{span}(\mathbf{A})/\text{span}(\mathbf{B}) \triangleq \{\mathbf{x} | \mathbf{x} \in \text{span}(\mathbf{A}), \mathbf{x} \notin \text{span}(\mathbf{B})\}$. $\mathbf{A} \succeq \mathbf{B}$ denotes that $\mathbf{A} - \mathbf{B}$ is a hermitian positive semidefinite matrix. $\mathbb{C}^{N \times M}$ indicates a $N \times M$ complex matrix set. $i \in \mathbb{Z}$ denotes that i is a positive integer. \mathbf{I} represents an identity matrix with appropriate size. Besides, $a^+ \triangleq \max(a, 0)$; $|a|$

is the magnitude of a ; $x \sim \mathcal{CN}(0, \Sigma)$ means x is a random variable following a complex circular Gaussian distribution with mean zero and covariance Σ .

II. SYSTEM MODEL AND PROBLEM STATEMENT

We consider the MIMO Gaussian wiretap channel with a cooperative jammer (see Fig.1) where the source, the legitimate receiver, the eavesdropper and the external helper are equipped with N_a , N_b , N_e and N_j antennas, respectively. The source wishes to send its message, \mathbf{x} , to the legitimate receiver, without being eavesdropped by the eavesdropper. Towards that objective, the source is aided by a cooperative terminal, which simultaneously transmits jamming signal, \mathbf{z} , to confuse the eavesdropper. The signals received at the legitimate receiver and the eavesdropper can be respectively expressed as

$$\mathbf{y}_d = \mathbf{H}_1 \mathbf{V} \mathbf{x} + \mathbf{G}_2 \mathbf{W} \mathbf{z} + \mathbf{n}_d \quad (1a)$$

$$\mathbf{y}_e = \mathbf{G}_1 \mathbf{V} \mathbf{x} + \mathbf{H}_2 \mathbf{W} \mathbf{z} + \mathbf{n}_e, \quad (1b)$$

where \mathbf{V} and \mathbf{W} are the precoding matrices at the source and the helper, respectively; $\mathbf{n}_d \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ and $\mathbf{n}_e \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ represent noise at the legitimate receiver and the eavesdropper, respectively; $\mathbf{G}_2 \in \mathbb{C}^{N_b \times N_j}$ and $\mathbf{H}_2 \in \mathbb{C}^{N_e \times N_j}$ represent the helper to legitimate receiver and the helper to eavesdropper channel matrices, respectively; $\mathbf{H}_1 \in \mathbb{C}^{N_b \times N_a}$ and $\mathbf{G}_1 \in \mathbb{C}^{N_e \times N_a}$ denote the channel matrix from the source to the legitimate receiver and the source to the eavesdropper, respectively. All channels are assumed to be flat fading. We assume that global channel state information (CSI) is available, including the CSI for the eavesdropper. This is possible in situations in which the eavesdropper is normally an active member of the network, communicating nonconfidential information with the other parties in other time slots [4]. A minimum-Mean-Square-Error (MMSE) receiver is considered at the legitimate receiver and the eavesdropper. The rate at the legitimate receiver and the eavesdropper can be respectively expressed as

$$R_d = \log |\mathbf{I} + (\mathbf{I} + \mathbf{G}_2 \mathbf{Q}_j \mathbf{G}_2^H)^{-1} \mathbf{H}_1 \mathbf{Q}_a \mathbf{H}_1^H| \quad (2a)$$

$$R_e = \log |\mathbf{I} + (\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H)^{-1} \mathbf{G}_1 \mathbf{Q}_a \mathbf{G}_1^H|, \quad (2b)$$

where $\mathbf{Q}_a \triangleq \mathbf{V} \mathbf{V}^H$ and $\mathbf{Q}_j \triangleq \mathbf{W} \mathbf{W}^H$ are the transmit covariance matrices at the source and the helper, respectively. In the paper, we focus on the SRM problem [23], i.e.,¹

$$C_s \triangleq \max_{\{\mathbf{Q}_a \succeq \mathbf{0}, \mathbf{Q}_j \succeq \mathbf{0}\}} R_d - R_e \quad \text{s.t.} \quad \text{tr}\{\mathbf{Q}_a + \mathbf{Q}_j\} \leq P, \quad (3)$$

where P is a given total transmit power budget and C_s denotes the maximal achievable secrecy rate, also known as the secrecy capacity.

Generally, the optimization problem of (3) is nonconvex. It is challenging and still an open problem to determine the exact secrecy capacity. In this paper, we propose to solve the problem of (3) by reformulating it into a sequence of convex problems. Also, we study the rate at which the secrecy capacity

¹For a given point $\{\mathbf{Q}_a, \mathbf{Q}_j\}$, the achieved secrecy rate is $\max(R_d - R_e, 0)$. For ease of exposition, the trivial case with zero achievable secrecy rate is omitted.

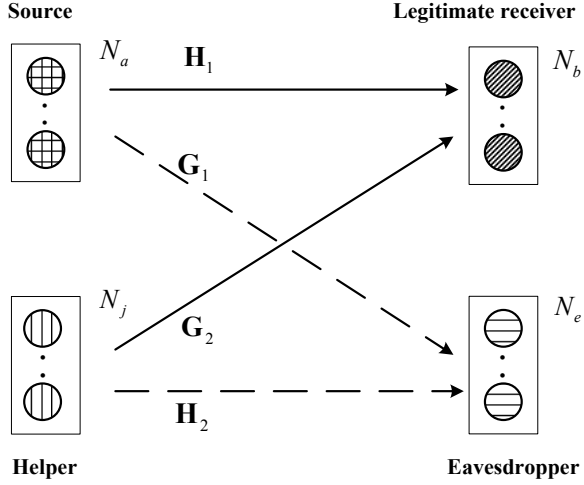


Fig. 1: MIMO wiretap channel with an external helper

scales with $\log(P)$, i.e., the maximal achievable s.d.o.f. [19], which equals

$$s.d.o.f. \triangleq \lim_{P \rightarrow \infty} \frac{C_s}{\log P}. \quad (4)$$

We compute *s.d.o.f.* analytically and determine its connection to system parameters, i.e., the number of antennas at each terminal.

In the following sections, we will begin with the simple case where the legitimate receiver is equipped with one antenna. Then, a more complicated scenario, in which each terminal is equipped with multiple antennas will be investigated.

III. HELPER-ASSISTED MISOME WIRETAP CHANNEL

In this section, we consider the helper-assisted multi-input single-output multi-antenna-eavesdropper (MISOME) wiretap channel where the legitimate receiver is equipped with a single antenna ($N_b = 1$). In such case, the legitimate receiver can receive at most one data stream. Thus, the source transmits one data stream x . Let \mathbf{v} denote the precoding vector at the source. The signals received at the legitimate receiver and the eavesdropper can be respectively expressed as

$$y_d = \mathbf{h}_1 \mathbf{v} x + \mathbf{g}_2 \mathbf{W} \mathbf{z} + n_d \quad (5)$$

$$\mathbf{y}_e = \mathbf{G}_1 \mathbf{v} x + \mathbf{H}_2 \mathbf{W} \mathbf{z} + \mathbf{n}_e, \quad (6)$$

where $\mathbf{h}_1 \in \mathbb{C}^{1 \times N_a}$ denotes the channel vector from the source to the legitimate receiver, and $\mathbf{g}_2 \in \mathbb{C}^{1 \times N_j}$ represents the channel vector from the helper to the legitimate receiver. The rate at the legitimate receiver and the eavesdropper can be simplified as,

$$R_d = \log(1 + (1 + \mathbf{g}_2 \mathbf{Q}_j \mathbf{g}_2^H)^{-1} \mathbf{h}_1 \mathbf{v} \mathbf{v}^H \mathbf{h}_1^H) \quad (7a)$$

$$R_e = \log(1 + \mathbf{v}^H \mathbf{G}_1^H (\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H)^{-1} \mathbf{G}_1 \mathbf{v}). \quad (7b)$$

Correspondingly, the secrecy capacity equals

$$C_s = \max_{\{\mathbf{v}, \mathbf{Q}_j \succeq \mathbf{0}\}} R_d - R_e \quad \text{s.t.} \quad \text{tr}\{\mathbf{v} \mathbf{v}^H + \mathbf{Q}_j\} \leq P. \quad (8)$$

Due to the presence of the multi-antenna eavesdropper, the SRM problem becomes more complex as compared with

the problem considered in [6]–[8], [10]. To cope with this issue, we resort to theorems on partial ordering of hermitian matrix [24]. Based on these theorems, we transform the original matrix inverse constraint into a convex linear matrix inequality (LMI) constraint, which, together with the semidefinite relaxation (SDR) technique, enables us to recast the original nonconvex optimization problem into a sequence of semidefinite programmes (SDPs). Further, we prove that the optimal solutions to the relaxed optimization problem are also optimal solutions to the original SRM problem. Consequently, we obtain the optimal solution to the original SRM problem with a combination of convex optimization and one-dimensional search. On the other hand, to gain more insight into how the secrecy capacity of the helper-assisted MISOME Gaussian wiretap channel behaves, we examine the conditions under which a s.d.o.f. equal to 1 can be achieved. To this end, we first introduce an alternative optimization problem which keeps the message signal and the jamming signal into different subspaces at the legitimate receiver, but aligns them into the same subspace at the eavesdropper. We then give two key lemmas. Lemma 1 proves that, a s.d.o.f. equal to 1 can be achieved if and only if the newly introduced optimization problem returns a nonempty set. Lemma 2 gives the conditions under which the newly introduced optimization problem returns a nonempty set. Combining the two lemmas, we finally show that a s.d.o.f. equal to 1 can be achieved if and only if $N_e < N_a + N_j - 1$.

A. Secrecy rate maximization

To solve the SRM problem in (8), the Two-Layer idea of [8] is adopted. The key insight is to recast the original optimization problem in (8) as a two-level optimization problem. The inner-level part is dealt with the SDR technique, and the outer-level part is handled by one-dimensional search. Specifically, the outer-level part is

$$\max_{\tau \in [\tau_{lb}, \tau_{ub}]} \log(1 + g(\tau)) - \log(1 + \tau), \quad (9)$$

where $g(\tau)$ is obtained by solving the following inner-level part optimization problem for a fixed τ :

$$g(\tau) = \max_{\{\mathbf{v}, \mathbf{Q}_j \succeq \mathbf{0}\}} \mathbf{h}_1 \mathbf{v} \mathbf{v}^H \mathbf{h}_1^H / (1 + \mathbf{g}_2 \mathbf{Q}_j \mathbf{g}_2^H) \quad (10a)$$

$$\text{s.t.} \quad \mathbf{v}^H \mathbf{G}_1^H (\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H)^{-1} \mathbf{G}_1 \mathbf{v} \leq \tau \quad (10b)$$

$$\text{tr}\{\mathbf{v} \mathbf{v}^H + \mathbf{Q}_j\} \leq P. \quad (10c)$$

By performing one-dimensional search on τ , the optimal τ^* maximizing the objective function in (9) can be found. Correspondingly, the optimal solution $\{\mathbf{v}^*, \mathbf{Q}_j^*\}$ to the original optimization problem of (8) can be obtained.

In (9), τ_{lb} and τ_{ub} denote the lower and upper bound on γ_e , respectively. Firstly, it is obvious that γ_e is no less than 0. Thus, we have $\tau_{lb} = 0$. Secondly, according to the security requirement, γ_e should be no more than γ_d . Further, γ_d is upper bounded by the maximal received signal-to-noise ratio (SNR) value of $P|\mathbf{h}_1|^2$ at the legitimate receiver. Therefore, $\tau_{ub} = P|\mathbf{h}_1|^2$.

So far, τ_{lb} and τ_{ub} have been determined. In the following, we focus on solving the optimization problem of (10), which

is still nonconvex. To solve it, we resort to the SDR technique of [25]. On denoting $\mathbf{Q}_a = \mathbf{v}\mathbf{v}^H$ and dropping the rank-one constraint, the optimization problem of (10) can be rewritten as

$$f(\tau) = \max_{\{\mathbf{Q}_a \succeq \mathbf{0}, \mathbf{Q}_j \succeq \mathbf{0}\}} \mathbf{h}_1 \mathbf{Q}_a \mathbf{h}_1^H / (1 + \mathbf{g}_2 \mathbf{Q}_j \mathbf{g}_2^H) \quad (11a)$$

$$\text{s.t. } \mathbf{G}_1 \mathbf{Q}_a \mathbf{G}_1^H \preceq \tau (\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H) \quad (11b)$$

$$\text{tr}\{\mathbf{Q}_a + \mathbf{Q}_j\} \leq P, \quad (11c)$$

where the replacement of the constraint (10b) with (11b) can be proven using basic theorems on partial ordering [24] as follows:

$$\begin{aligned} (10b) &\Leftrightarrow \lambda_{\max}((\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H)^{-1} \mathbf{G}_1 \mathbf{v} \mathbf{v}^H \mathbf{G}_1^H) \leq \tau \\ &\Leftrightarrow (\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H)^{-1} \mathbf{G}_1 \mathbf{v} \mathbf{v}^H \mathbf{G}_1^H \preceq \tau \mathbf{I} \\ &\Leftrightarrow \mathbf{G}_1 \mathbf{v} \mathbf{v}^H \mathbf{G}_1^H \preceq \tau (\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H) \Leftrightarrow (11b). \end{aligned}$$

In the above, $\lambda_{\max}(\mathbf{A})$ denotes the maximum eigenvalue of \mathbf{A} .

Letting $\xi = (1 + \mathbf{g}_2 \mathbf{Q}_j \mathbf{g}_2^H)^{-1} > 0$, $\tilde{\mathbf{Q}}_a = \xi \mathbf{Q}_a$, $\tilde{\mathbf{Q}}_j = \xi \mathbf{Q}_j$, and using the Charnes-Cooper transformation [26], we can recast the optimization problem of (11) as

$$\begin{aligned} f(\tau) &= \max_{\{\tilde{\mathbf{Q}}_a \succeq \mathbf{0}, \tilde{\mathbf{Q}}_j \succeq \mathbf{0}, \xi > 0\}} \mathbf{h}_1 \tilde{\mathbf{Q}}_a \mathbf{h}_1^H \\ \text{s.t. } &\xi + \mathbf{g}_2 \tilde{\mathbf{Q}}_j \mathbf{g}_2^H = 1 \\ &\mathbf{G}_1 \tilde{\mathbf{Q}}_a \mathbf{G}_1^H \preceq \tau (\xi \mathbf{I} + \mathbf{H}_2 \tilde{\mathbf{Q}}_j \mathbf{H}_2^H) \\ &\text{tr}\{\tilde{\mathbf{Q}}_a + \tilde{\mathbf{Q}}_j\} \leq \xi P, \end{aligned} \quad (12)$$

which is a SDP and can be efficiently solved using available software packages, e.g., CVX [26].

Let us consider the power minimization problem associated with (11), which can be formulated as follows:

$$\begin{aligned} \min_{\{\mathbf{Q}_a, \mathbf{Q}_j\}} &\text{tr}\{\mathbf{Q}_a + \mathbf{Q}_j\} \\ \text{s.t. } &\mathbf{h}_1 \mathbf{Q}_a \mathbf{h}_1^H / (1 + \mathbf{g}_2 \mathbf{Q}_j \mathbf{g}_2^H) \geq f(\tau) \\ &\mathbf{G}_1 \mathbf{Q}_a \mathbf{G}_1^H \preceq \tau (\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H) \\ &\mathbf{Q}_a \succeq \mathbf{0}, \mathbf{Q}_j \succeq \mathbf{0}, \end{aligned} \quad (13)$$

where $f(\tau)$ is obtained by solving the optimization problem of (12). We have the following two propositions.

Proposition 1: Denote the optimal solution to (13) as $\{\hat{\mathbf{Q}}_a, \hat{\mathbf{Q}}_j\}$. Then, $\hat{\mathbf{Q}}_a$ is rank-one provided that a positive secrecy rate is achieved.

Proof: See Appendix A. ■

Proposition 2: Denote the optimal solution to (13) as $\{\hat{\mathbf{Q}}_a, \hat{\mathbf{Q}}_j\}$. Then, $\{\hat{\mathbf{Q}}_a, \hat{\mathbf{Q}}_j\}$ is also the optimal solution to the problem of (11).

Proof: See Appendix B. ■

Let $\mathbf{Q}_a^o = \hat{\mathbf{Q}}_a$ and $\mathbf{Q}_j^o = \hat{\mathbf{Q}}_j$. Combining Proposition 1 with Proposition 2, we get that $\{\mathbf{Q}_a^o, \mathbf{Q}_j^o\}$ is the optimal solution to the problem of (11), such that $\text{rank}\{\mathbf{Q}_a^o\} = 1$. Therefore, the optimization problem of (11) is indeed a tight approximation of the optimization problem of (10). Moreover, $\{\mathbf{Q}_a^o, \mathbf{Q}_j^o\}$ is also the optimal solution to the problem of (10) and $g(\tau) = f(\tau)$.

B. Conditions to ensure s.d.o.f. equal to 1

As stated in the preceding sections, it is difficult to obtain an analytical expression for the secrecy capacity for the helper-assisted Gaussian wiretap channel. Instead, in this subsection, we investigate the conditions under which a s.d.o.f. equal to 1 can be achieved. To this end, we first introduce an alternative optimization problem as follows:

$$\text{find } \{\mathbf{v}, \mathbf{W}\} \quad (14a)$$

$$\text{s.t. } \text{span}(\mathbf{G}_1 \mathbf{v}) \subset \text{span}(\mathbf{H}_2 \mathbf{W}) \quad (14b)$$

$$\text{span}(\mathbf{g}_2 \mathbf{W}) \cap \text{span}(\mathbf{h}_1 \mathbf{v}) = \{0\} \quad (14c)$$

$$|\mathbf{h}_1 \mathbf{v}| > 0. \quad (14d)$$

Specifically, we aim to find the point at which the subspace spanned by the message signal and that spanned by the jamming signal have no intersection at the legitimate receiver, such that R_d scales with $\log(P)$. Simultaneously, the subspace spanned by the message signal belongs to the subspace spanned by the jamming signal at the eavesdropper, such that R_e converges to a constant as P approaches to infinity.

In the sequel, we first give two key lemmas. Lemma 1 proves that s.d.o.f. equal to 1 can be achieved if and only if the optimization problem of (14) returns a nonempty set. Lemma 2 gives the conditions under which the optimization problem of (14) returns a nonempty set. Combining the two lemmas, we finally obtain the conditions to ensure s.d.o.f equal to 1 in the helper-assisted MISOME Gaussian wiretap channel.

Lemma 1: The secure degrees of freedom equal to 1 can be achieved if and only if the optimization problem of (14) returns a nonempty set.

Proof: Clearly, if the optimization problem of (14) returns a nonempty set, then s.d.o.f. equal to 1 can be achieved. So the sufficiency holds true.

We now turn to prove the necessity by contradiction. If the optimization problem of (14) returns an empty set, then at least one of the constraints in (14) does not hold true. We test (14b)-(14d) one by one:

- 1) If (14b) does not hold true, then there exists a direction along which the eavesdropper can extract the message signal without interference, so the rate at which R_e scales with $\log(P)$ is 1. Together with (4),(8) and the fact that the rate at which R_d scales with $\log(P)$ is at most 1 for the multi-input single-output (MISO) source-receiver channel, we arrive at $s.d.o.f. = 0$.
- 2) If (14c) does not hold true, then the message signal is aligned in the subspace spanned by the jamming signal, so R_d converges to a constant when P approaches to infinity, which indicates that $s.d.o.f. = 0$.
- 3) If (14d) does not hold true, then $|\mathbf{h}_1 \mathbf{v}| = 0$, which indicates that $R_d = 0$, thus $s.d.o.f. = 0$.

Summarizing, if the optimization problem of (14) returns an empty set, $s.d.o.f. = 0$. Therefore, if $s.d.o.f. = 1$, the optimization problem of (14) returns a nonempty set. This completes the proof. ■

Before proceeding to Lemma 2, we first introduce the generalized singular value decomposition (GSVD) transform, which provides the basis for the proof of Lemma 2 to follow.

Definition 1 (GSVD Transform): Given two matrices $\mathbf{H} \in \mathbb{C}^{N \times M}$ and $\mathbf{G} \in \mathbb{C}^{N \times K}$, let

$$k \triangleq \text{rank}\{[\mathbf{H}^H, \mathbf{G}^H]^T\} \quad (15a)$$

$$p \triangleq \dim\{\text{span}(\mathbf{H})^\perp \cap \text{span}(\mathbf{G})\} \quad (15b)$$

$$r \triangleq \dim\{\text{span}(\mathbf{H}) \cap \text{span}(\mathbf{G})^\perp\} \quad (15c)$$

$$s \triangleq \dim\{\text{span}(\mathbf{H}) \cap \text{span}(\mathbf{G})\}, \quad (15d)$$

then we have

$$k = \min\{M + K, N\} \quad (16a)$$

$$p = k - \min\{M, N\} \quad (16b)$$

$$r = k - \min\{K, N\} \quad (16c)$$

$$s = \min\{M, N\} + \min\{K, N\} - k. \quad (16d)$$

The proof is given in Appendix C. According to [27], the GSVD of $(\mathbf{H}^H, \mathbf{G}^H)$ returns unitary matrices $\Psi_1 \in \mathbb{C}^{M \times M}$ and $\Psi_2 \in \mathbb{C}^{K \times K}$, non-negative diagonal matrices $\mathbf{D}_1 \in \mathbb{C}^{M \times k}$ and $\mathbf{D}_2 \in \mathbb{C}^{K \times k}$, and a matrix $\mathbf{X} \in \mathbb{C}^{N \times k}$ with $\text{rank}\{\mathbf{X}\} = k$, such that

$$\mathbf{H}\Psi_1 = \mathbf{X}\mathbf{D}_1^H \quad (17a)$$

$$\mathbf{G}\Psi_2 = \mathbf{X}\mathbf{D}_2^H, \quad (17b)$$

$$\text{in which } \mathbf{D}_1 = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathbf{D}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_p \end{bmatrix},$$

where the diagonal entries of $\mathbf{S}_1 \in \mathbb{R}^{s \times s}$ and $\mathbf{S}_2 \in \mathbb{R}^{s \times s}$ are greater than 0, and $\mathbf{D}_1^H \mathbf{D}_1 + \mathbf{D}_2^H \mathbf{D}_2 = \mathbf{I}$.

For simplicity, in the following text of this paper, we denote the above GSVD Transform as

$$(\Psi_1, \Psi_2, \mathbf{D}_1, \mathbf{D}_2, \mathbf{X}, k, r, s, p) = \text{gsvd}(\mathbf{H}^H, \mathbf{G}^H).$$

Lemma 2: The optimization problem of (14) returns a nonempty set if and only if $N_e < N_a + N_j - 1$.

Proof: We start with the constraint of (14c). With the GSVD Transform of $(\mathbf{h}_1^H, \mathbf{g}_2^H)$, we get $s_1 \triangleq \dim\{\text{span}(\mathbf{h}_1) \cap \text{span}(\mathbf{g}_2)\} = 1$. Thus to satisfy (14c), we should have $|\mathbf{h}_1 \mathbf{v}| = 0$ or $|\mathbf{g}_2 \mathbf{W}| = 0$. However, $|\mathbf{h}_1 \mathbf{v}| = 0$ contradicts (14d), so we should have $|\mathbf{g}_2 \mathbf{W}| = 0$.

Without loss of generality, let $\mathbf{W} = \Gamma \mathbf{W}_1$ where $\Gamma = \text{null}\{\mathbf{g}_2\} \in \mathbb{C}^{N_j \times (N_j - 1)}$, $\mathbf{W}_1 \in \mathbb{C}^{(N_j - 1) \times (N_j - 1)}$. Substituting $\mathbf{W} = \Gamma \mathbf{W}_1$ into (14b), we arrive at

$$\text{span}(\mathbf{G}_1 \mathbf{v}) \subset \text{span}(\mathbf{H}_2 \Gamma \mathbf{W}_1), \quad (18)$$

in which $\mathbf{G}_1 \in \mathbb{C}^{N_e \times N_a}$ and $\bar{\mathbf{H}}_2 \triangleq \mathbf{H}_2 \Gamma \in \mathbb{C}^{N_e \times (N_j - 1)}$.

Invoking the GSVD Transform of $(\bar{\mathbf{H}}_2^H, \mathbf{G}_1^H)$, we obtain

$$(\bar{\Psi}_1, \bar{\Psi}_2, \bar{\mathbf{D}}_1, \bar{\mathbf{D}}_2, \bar{\mathbf{X}}, k_2, r_2, s_2, p_2) = \text{gsvd}(\bar{\mathbf{H}}_2^H, \mathbf{G}_1^H),$$

such that

$$\bar{\mathbf{H}}_2 \bar{\Psi}_1 = \bar{\mathbf{X}} \bar{\mathbf{D}}_1^H \quad (19a)$$

$$\mathbf{G}_1 \bar{\Psi}_2 = \bar{\mathbf{X}} \bar{\mathbf{D}}_2^H, \quad (19b)$$

where $k_2 = \min\{N_a + N_j - 1, N_e\}$, $p_2 = k_2 - \min\{N_j - 1, N_e\}$, $r_2 = k_2 - \min\{N_a, N_e\}$ and $s_2 = \min\{N_a, N_e\} + \min\{N_j - 1, N_e\} - k_2$.

To satisfy (14d), we should have $\mathbf{v} \neq \mathbf{0}$, which, together with (19a) and (19b), indicates that (18) holds true if and only if $p_2 < N_a$.

1) For the case of $N_e \leq N_j - 1$, $p_2 = N_e - N_e = 0$. So $p_2 < N_a$.

2) For the case of $N_j - 1 < N_e < N_a + N_j - 1$, $p_2 = N_e - (N_j - 1) < N_a$.

3) For the case of $N_e \geq N_a + N_j - 1$, $p_2 = N_a + N_j - 1 - (N_j - 1) = N_a$.

Summarizing, $p_2 < N_a$ if and only if $N_e < N_a + N_j - 1$.

Therefore, if $N_e < N_a + N_j - 1$, (18) holds true, thus the optimization problem of (14) returns a nonempty set. Otherwise, if $N_e \geq N_a + N_j - 1$, (18) does not hold true, so the optimization problem of (14) returns an empty set. In a word, the optimization problem of (14) returns a nonempty set if and only if $N_e < N_a + N_j - 1$. This completes the proof. ■

Thorem 1: The secure degrees of freedom equal to 1 can be achieved if and only if $N_e < N_a + N_j - 1$.

Proof: Combining Lemma 1 with Lemma 2, it is clear that the secure degrees of freedom equal to 1 can be achieved if and only if $N_e < N_a + N_j - 1$. This completes the proof. ■

Remark: It is worthwhile to note that Lemma 2 also provides us with a way to determine the precoding matrices at the source and the helper to achieve s.d.o.f. of 1. In the remaining text of this subsection, we first give the precoding matrices to achieve s.d.o.f. of 1 in closed-form. We then substitute the derived precoding matrices into (8) and solve for the optimal power allocation between the message signal and the jamming signal. Consequently, we obtain an analytical lower bound on the secrecy capacity.

Revisiting the proof of Lemma 2, for the case of $N_e < N_a + N_j - 1$, $k_2 = N_e$, $r_2 = k_2 - \min\{N_a, N_e\}$ and $s_2 = \min\{N_a, N_e\} + \min\{N_j - 1, N_e\} - k_2$. Actually, $s_2 > 0$ which can be justified as follows: (i) For the case of $N_e \leq N_j - 1$, $s_2 = \min\{N_a, N_e\} > 0$; (ii) For the case of $N_j - 1 < N_e < N_a + N_j - 1$, $s_2 = \min\{N_a, N_e\} - (N_j - 1) - N_e > 0 \Leftrightarrow \min\{N_a, N_e\} > N_e - (N_j - 1)$ where the latter inequality can be easily verified.

Let $\mathcal{I} = \{j | r_2 + 1 \leq j \leq r_2 + s_2, j \in \mathbb{Z}\}$. Since $s_2 > 0$, \mathcal{I} is a nonempty set. Let $\mathbf{W}_1^o = \bar{\Psi}_1(:, i) / |\bar{\Psi}_1(:, i)|$ and $\mathbf{v}_o = \bar{\Psi}_2(:, i + N_a - k_2) / |\bar{\Psi}_2(:, i + N_a - k_2)|$ where $i \in \mathcal{I}$. According to (19a) and (19b), we arrive at $\text{span}(\mathbf{G}_1 \mathbf{v}_o) = \text{span}(\mathbf{H}_2 \Gamma \mathbf{W}_1^o) = \text{span}(\bar{\mathbf{X}}(:, i))$, so $\{\mathbf{v}_o, \Gamma \mathbf{W}_1^o\}$ is a feasible point to (14). Substituting $\mathbf{v} = \sqrt{x} \mathbf{v}_o$ and $\mathbf{W} = \sqrt{P - x} \Gamma \mathbf{W}_1^o$ into (8), $0 \leq x \leq P$, we arrive at

$$C_s^{\text{sub}} \triangleq \max_{\{0 \leq x \leq P\}} \log \frac{1 + |\mathbf{h}_1 \mathbf{v}_o|^2 x}{1 + \gamma_e^{\text{sub}}}, \quad (20)$$

in which

$$\begin{aligned} \gamma_e^{\text{sub}} &= x \mathbf{v}_o^H \mathbf{G}_1^H (\mathbf{I} + (P - x) \bar{\mathbf{H}}_2 \mathbf{W}_1^o \mathbf{W}_1^{oH} \bar{\mathbf{H}}_2^H)^{-1} \mathbf{G}_1 \mathbf{v}_o \\ &= x |\mathbf{G}_1 \mathbf{v}_o|^2 - \frac{x(P - x) |\mathbf{W}_1^{oH} \bar{\mathbf{H}}_2^H \mathbf{G}_1 \mathbf{v}_o|^2}{1 + (P - x) |\bar{\mathbf{H}}_2 \mathbf{W}_1^o|^2} \end{aligned} \quad (21a)$$

$$= x |\mathbf{G}_1 \mathbf{v}_o|^2 - \frac{x(P - x) |\bar{\mathbf{H}}_2 \mathbf{W}_1^o|^2 |\mathbf{G}_1 \mathbf{v}_o|^2}{1 + (P - x) |\bar{\mathbf{H}}_2 \mathbf{W}_1^o|^2}, \quad (21b)$$

where (21a) follows from the matrix inverse lemma [24], and (21b) follows from the fact that $\text{span}(\mathbf{G}_1 \mathbf{v}_o) = \text{span}(\bar{\mathbf{H}}_2 \mathbf{W}_1^o) = \text{span}(\bar{\mathbf{X}}(:, i))$.

For ease of exposition, let $a = |\mathbf{h}_1 \mathbf{v}_o|^2$, $b = |\mathbf{G}_1 \mathbf{v}_o|^2$ and $c = |\bar{\mathbf{H}}_2 \mathbf{W}_1^o|^2$. Also, noting that the logarithm function is a monotonically increasing function, therefore the optimization problem of (20) becomes

$$2^{C_s^{\text{sub}}} = \max_{0 \leq x \leq P} \eta(x), \quad (22)$$

in which

$$\begin{aligned} \eta(x) &\triangleq \frac{1 + ax}{1 + bx - [bc(P - x)x / (1 + c(P - x))]} \\ &= \frac{(1 + ax)[1 + c(P - x)]}{1 + cP + (b - c)x}. \end{aligned} \quad (23)$$

Resorting to carefully mathematical deductions, we solve (22) and arrive at that when the total transmit power P is big enough,

$$C_s^{\text{sub}} \approx \log(aP) - 2\log(1 + \sqrt{b/c}), \quad (24)$$

where the details are given in Appendix D.

Substituting $a = |\mathbf{h}_1 \mathbf{v}_o|^2$, $b = |\mathbf{G}_1 \mathbf{v}_o|^2$ and $c = |\bar{\mathbf{H}}_2 \mathbf{W}_1^o|^2$ into (24) yields

$$C_s^{\text{sub}} \approx \log(|\mathbf{h}_1 \mathbf{v}_o|^2 P) - 2\log(1 + \sqrt{|\mathbf{G}_1 \mathbf{v}_o|^2 / |\bar{\mathbf{H}}_2 \mathbf{W}_1^o|^2}), \quad (25)$$

which in turn explicitly corroborates that s.d.o.f. equal to 1 has been achieved.

IV. HELPER-ASSISTED MIMOME WIRETAP CHANNEL

In this section, we consider the helper-assisted MIMOME wiretap channel where each terminal is equipped with multiple antennas. In such MIMO case, the SRM problem becomes more complex as compared with the problem considered in Section III, and actually, it is still an open problem. To deal with this issue, we first reformulate the SRM problem in (3) to a form that can be processed with the Gauss-Seidel approach, which successively optimizes each variable given that the other variables are fixed, thus giving an iterative algorithm to solve (3). We then examine the maximal achievable s.d.o.f. and reveal its connection to the number of antennas at each terminal. We obtain both the maximal achievable s.d.o.f. and the solution that achieves the maximal s.d.o.f. in closed-form.

A. Secrecy rate maximization

In order to reformulate the SRM problem in (3) to a form that can be processed with the Gauss-Seidel approach, we need the following lemma.

Lemma 3: Given a positive definite matrix $\mathbf{E} \in \mathbb{C}^{N \times N}$, it holds that

$$\ln |\mathbf{E}^{-1}| = \max_{\mathbf{S} \in \mathbb{N} \times \mathbb{N}, \mathbf{S} \geq \mathbf{0}} \varphi(\mathbf{S}), \quad (26)$$

where $\varphi(\mathbf{S}) = -\text{tr}(\mathbf{S}\mathbf{E}) + \ln |\mathbf{S}| + N$. Moreover, for the optimal solution to the right-hand side of (26), it holds that $\mathbf{S}^* = \mathbf{E}^{-1}$.

Proof: Please refer to [28]. ■

Applying Lemma 3, we arrive at, respectively,

$$\ln |(\mathbf{I} + \mathbf{G}_2 \mathbf{Q}_j \mathbf{G}_2^H)^{-1}| = \max_{\mathbf{S}_0 \geq \mathbf{0}} \varphi_b(\mathbf{S}_0) \quad (27a)$$

$$\ln |(\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H + \mathbf{G}_1 \mathbf{Q}_a \mathbf{G}_1^H)^{-1}| = \max_{\mathbf{S}_1 \geq \mathbf{0}} \varphi_e(\mathbf{S}_1), \quad (27b)$$

where $\varphi_b(\mathbf{S}_0) = -\text{tr}\{\mathbf{S}_0(\mathbf{I} + \mathbf{G}_2 \mathbf{Q}_j \mathbf{G}_2^H)\} + \ln |\mathbf{S}_0| + N_b$, and $\varphi_e(\mathbf{S}_1) = -\text{tr}\{\mathbf{S}_1(\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H + \mathbf{G}_1 \mathbf{Q}_a \mathbf{G}_1^H)\} + \ln |\mathbf{S}_1| + N_e$.

Substituting (27a) and (27b) into (2a) and (2b), respectively, we arrive at

$$R_d = \max_{\mathbf{S}_0 \geq \mathbf{0}} \varphi_b(\mathbf{S}_0) + \ln |\mathbf{I} + \mathbf{H}_1 \mathbf{Q}_a \mathbf{H}_1^H + \mathbf{G}_2 \mathbf{Q}_j \mathbf{G}_2^H| \quad (28a)$$

$$R_e = -\max_{\mathbf{S}_1 \geq \mathbf{0}} \varphi_e(\mathbf{S}_1) - \ln |\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H|. \quad (28b)$$

Further, substituting (28a)(28b) into (3), we arrive at

$$\begin{aligned} C_s &= \max_{\{\mathbf{Q}_a \geq \mathbf{0}, \mathbf{Q}_j \geq \mathbf{0}, \mathbf{S}_0 \geq \mathbf{0}, \mathbf{S}_1 \geq \mathbf{0}\}} \theta(\mathbf{S}_0, \mathbf{S}_1, \mathbf{Q}_a, \mathbf{Q}_j) \\ \text{s.t. } &\text{tr}\{\mathbf{Q}_a + \mathbf{Q}_j\} \leq P, \end{aligned} \quad (29)$$

where $\theta(\mathbf{S}_0, \mathbf{S}_1, \mathbf{Q}_a, \mathbf{Q}_j) = \varphi(\mathbf{S}_0) + \varphi_e(\mathbf{S}_1) + \omega(\mathbf{Q}_a, \mathbf{Q}_j)$ in which $\omega(\mathbf{Q}_a, \mathbf{Q}_j) = \ln |\mathbf{I} + \mathbf{H}_1 \mathbf{Q}_a \mathbf{H}_1^H + \mathbf{G}_2 \mathbf{Q}_j \mathbf{G}_2^H| + \ln |\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H|$.

Although the optimization problem of (29) is still not convex, we observe that if we fix either $\{\mathbf{Q}_a, \mathbf{Q}_j\}$ or $\mathbf{S}_i (i = 1, 2)$, the remaining problem is convex and can thus be solved efficiently. Hence, we turn to a two-stage iterative method (Gauss-Seidel approach), and approximately solve the optimization problem of (29) via iterations between the following two subproblems.

- 1) Fix $\{\mathbf{Q}_a, \mathbf{Q}_j\}$, and maximize $\theta(\mathbf{S}_0, \mathbf{S}_1, \mathbf{Q}_a, \mathbf{Q}_j)$ over $\{\mathbf{S}_0, \mathbf{S}_1\}$.
- 2) Fix $\{\mathbf{S}_0, \mathbf{S}_1\}$, and maximize $\theta(\mathbf{S}_0, \mathbf{S}_1, \mathbf{Q}_a, \mathbf{Q}_j)$ over $\{\mathbf{Q}_a, \mathbf{Q}_j\}$.

Specifically, when $\{\mathbf{Q}_a, \mathbf{Q}_j\}$ is fixed, let

$$\{\mathbf{S}_0^*, \mathbf{S}_1^*\} = \arg \max_{\{\mathbf{S}_0, \mathbf{S}_1\}} \theta(\mathbf{S}_0, \mathbf{S}_1, \mathbf{Q}_a, \mathbf{Q}_j).$$

Applying Lemma 3, we arrive at

$$\begin{aligned} \mathbf{S}_0^* &= (\mathbf{I} + \mathbf{G}_2 \mathbf{Q}_j \mathbf{G}_2^H)^{-1}, \\ \mathbf{S}_1^* &= (\mathbf{I} + \mathbf{H}_2 \mathbf{Q}_j \mathbf{H}_2^H + \mathbf{G}_1 \mathbf{Q}_a \mathbf{G}_1^H)^{-1}. \end{aligned}$$

Besides, when $\{\mathbf{S}_0, \mathbf{S}_1\}$ is fixed, the maximization of $\theta(\mathbf{S}_0, \mathbf{S}_1, \mathbf{Q}_a, \mathbf{Q}_j)$ over $\{\mathbf{Q}_a, \mathbf{Q}_j\}$ is a convex optimization problem and can be efficiently solved by available software packages, e.g., CVX [26].

One can easily verify that the above iterative process leads to a monotonically non-descending objective function value of $\theta(\mathbf{S}_0, \mathbf{S}_1, \mathbf{Q}_a, \mathbf{Q}_j)$. Moreover, for a given limited transmit power, the achievable secrecy rate is upper bounded. Thus, the above iterative algorithm is convergent.

Remark: Although the above iterative algorithm provides no guarantee of finding the global optimal solution to the problem of (29), our numerical results in the following section show that it attains a fairly good secrecy rate performance.

B. Maximal achievable secure degrees of freedom

In this subsection, we examine the maximal achievable s.d.o.f. and determine its connection to the number of antennas at each terminal. Similar to Section III, we first introduce an alternative optimization problem as follows:

$$d \triangleq \max_{\{\mathbf{V}, \mathbf{W}\}} \text{rank}\{\mathbf{H}_1 \mathbf{V}\} \quad (30a)$$

$$\text{s.t. } \text{span}(\mathbf{G}_1 \mathbf{V}) \subset \text{span}(\mathbf{H}_2 \mathbf{W}) \quad (30b)$$

$$\text{span}(\mathbf{G}_2 \mathbf{W}) \cap \text{span}(\mathbf{H}_1 \mathbf{V}) = \{\mathbf{0}\}. \quad (30c)$$

Specifically, we find the feasible points at which the subspace spanned by the message signal and that spanned by the jamming signal have no intersection at the legitimate receiver. Simultaneously, the subspace spanned by the message signal belongs to the subspace spanned by the jamming signal at the eavesdropper. Among these feasible points, we determine the one at which the rank of $\mathbf{H}_1 \mathbf{V}$ is maximized.

Lemma 4: The maximal achievable secure degrees of freedom, defined in (4), equal to d . That is, s.d.o.f. = d .

Proof: See Appendix E ■

From Lemma 4, we observe that to obtain the s.d.o.f, we need only to focus on solving (30). To this end, in the sequel, we first give a heuristic method which gives a closed-form feasible point $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ to (30), followed by the derivation of $d^* \triangleq \text{rank}\{\mathbf{H}_1 \hat{\mathbf{V}}\}$ in closed-form. Subsequently, in Lemma 5, we prove that $d = d^*$. Combining Lemma 4 and Lemma 5, we finally obtain s.d.o.f. = d^* . Further, we prove that $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ achieves the maximal s.d.o.f., i.e., $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ constituting the s.d.o.f.-optimal solution to the original SRM problem.

The aforementioned heuristic method to obtain $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ is shown in Table I. Notice that in Table I, $\text{null}\{\mathbf{G}_1\}$ returns an empty matrix when $N_a \leq N_e$. In the following text, we prove that $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ is a feasible solution for (30), and derive the closed-form expression for d^* . As in Table I, four cases are discussed.

In *Case I* and *Case II*, it is clear that $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ is feasible to (30) and $d^* = \text{rank}\{\mathbf{H}_1 \hat{\mathbf{V}}\} = \min\{N_a, N_b\}$.

In *Case III*, for the subcase of $d_0 + d_1 \geq N_b$, $\hat{\mathbf{V}} = [\mathbf{V}_0, \mathbf{V}_1]$ and $\hat{\mathbf{W}} = \mathbf{W}_1$. According to (31), we get $\text{span}(\mathbf{G}_2 \mathbf{W}_1) = \{\mathbf{0}\}$ and $\text{span}(\mathbf{H}_2 \mathbf{W}_1) = \text{span}(\mathbf{G}_1 \mathbf{V}_1)$. In addition, $\mathbf{G}_1 \mathbf{V}_0 = \mathbf{0}$. So, $\text{span}(\mathbf{H}_2 \hat{\mathbf{W}}) = \text{span}(\mathbf{G}_1 \hat{\mathbf{V}})$ and $\text{span}(\mathbf{H}_1 \hat{\mathbf{V}}) \cap \text{span}(\mathbf{G}_2 \hat{\mathbf{W}}) = \{\mathbf{0}\}$, which indicate that $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ is feasible to (30). Furthermore, \mathbf{V}_0 is orthogonal with \mathbf{V}_1 by definition, thus

$$\begin{aligned} d^* &= \text{rank}\{[\mathbf{V}_0, \mathbf{V}_1]\} \\ &= \text{rank}\{\mathbf{V}_0\} + \text{rank}\{\mathbf{V}_1\} = N_b. \end{aligned}$$

For the subcase of $d_0 + d_1 < N_b$, $\hat{\mathbf{V}} = [\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2]$ and $\hat{\mathbf{W}} = [\mathbf{W}_1, \mathbf{W}_2]$. As in the subcase of $d_0 + d_1 \geq N_b$, $\mathbf{G}_1 \mathbf{V}_0 = \mathbf{0}$, $\text{span}(\mathbf{G}_2 \mathbf{W}_1) = \{\mathbf{0}\}$ and $\text{span}(\mathbf{H}_2 \mathbf{W}_1) = \text{span}(\mathbf{G}_1 \mathbf{V}_1)$. In addition, according to (32), $\text{span}(\mathbf{H}_2 \mathbf{W}_2) = \text{span}(\mathbf{G}_1 \mathbf{V}_2)$ and $d_2 = \min\{s_4, \lfloor \frac{N_b - (d_0 + d_1)}{2} \rfloor\}$. Thus $\text{span}(\mathbf{H}_2 \hat{\mathbf{W}}) = \text{span}(\mathbf{G}_1 \hat{\mathbf{V}})$ and $\text{span}(\mathbf{H}_1 \hat{\mathbf{V}}) \cap \text{span}(\mathbf{G}_2 \hat{\mathbf{W}}) = \{\mathbf{0}\}$. Therefore $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ is feasible to (30). Furthermore, $[\mathbf{V}_0, \mathbf{V}_1]$ is

TABLE I: A heuristic method to obtain $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ which is feasible to (30)

Case I: $N_a \geq N_e + N_b$. Let $\hat{\mathbf{V}} = \text{null}\{\mathbf{G}_1\}$ and $\hat{\mathbf{W}} = \mathbf{0}$.

Case II: $N_j \geq N_b + N_e$. Let $\hat{\mathbf{W}} = \text{null}\{\mathbf{G}_2\} \in \mathbb{C}^{N_a \times (N_j - N_b)}$ and $\hat{\mathbf{V}}$ be the right singular matrix of \mathbf{H}_1 .

Case III: $N_a < N_e + N_b$ and $N_b < N_j < N_e + N_b$. For a start, let $\mathbf{V}_0 = \text{null}\{\mathbf{G}_1\}$ and $d_0 = (N_a - N_e)^+$. Secondly, denote $\tilde{\mathbf{H}}_2 = \mathbf{H}_2 \mathbf{\Gamma} \in \mathbb{C}^{N_e \times (N_j - N_b)}$ where $\mathbf{\Gamma} = \text{null}\{\mathbf{G}_2\} \in \mathbb{C}^{N_a \times (N_j - N_b)}$. Denote $\tilde{\mathbf{G}}_1 = \mathbf{G}_1 \mathbf{V}_0^c$ where $\mathbf{V}_0^c = \text{null}\{\mathbf{V}_0^H\} \in \mathbb{C}^{N_a \times N_e}$. Invoking the GSVD Transform of $(\tilde{\mathbf{H}}_2^H, \tilde{\mathbf{G}}_1^H)$ yields

$$(\Psi_1, \Psi_2, \mathbf{D}_1, \mathbf{D}_2, \mathbf{X}, k_3, r_3, s_3) = \text{gsvd}(\tilde{\mathbf{H}}_2^H, \tilde{\mathbf{G}}_1^H). \quad (31)$$

Subsequently, let $d_1 = s_3$, $c_3 = r_3 + N_e - k_3$, and check

- 1) If $d_0 + d_1 \geq N_b$, let $\mathbf{W}_1 = \mathbf{\Gamma} \Psi_1(:, r_3 + 1 : r_3 + N_b - d_0)$ and $\mathbf{V}_1 = \mathbf{V}_0^c \Psi_2(:, c_3 + 1 : c_3 + N_b - d_0)$. Lastly, let $\hat{\mathbf{V}} = [\mathbf{V}_0, \mathbf{V}_1]$ and $\hat{\mathbf{W}} = \mathbf{W}_1$.
- 2) Otherwise, let $\mathbf{W}_1 = \mathbf{\Gamma} \Psi_1(:, r_3 + 1 : r_3 + s_3)$ and $\mathbf{V}_1 = \mathbf{V}_0^c \Psi_2(:, c_3 + 1 : c_3 + s_3)$. Thirdly, denote $\mathbf{V}_{01}^c = \text{null}\{[\mathbf{V}_0, \mathbf{V}_1]^H\} \in \mathbb{C}^{N_a \times (N_a - d_0 - d_1)}$ and $\tilde{\mathbf{G}}_1 = \mathbf{G}_1 \mathbf{V}_{01}^c$. Invoking the GSVD Transform of $(\mathbf{H}_2^H, \tilde{\mathbf{G}}_1^H)$ yields

$$(\tilde{\Psi}_1, \tilde{\Psi}_2, \tilde{\mathbf{D}}_1, \tilde{\mathbf{D}}_2, \tilde{\mathbf{X}}, k_4, r_4, s_4) = \text{gsvd}(\mathbf{H}_2^H, \tilde{\mathbf{G}}_1^H). \quad (32)$$

Then let $\mathbf{W}_2 = \tilde{\Psi}_1(:, r_4 + 1 : r_4 + d_2)$ and $\mathbf{V}_2 = \mathbf{V}_{01}^c \tilde{\Psi}_2(:, c_4 + 1 : c_4 + d_2)$ in which $d_2 = \min\{s_4, \lfloor \frac{N_b - (d_0 + d_1)}{2} \rfloor\}$ and $c_4 = r_4 + (N_a - d_0 - d_1) - k_4$. Lastly, let $\hat{\mathbf{V}} = [\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2]$ and $\hat{\mathbf{W}} = [\mathbf{W}_1, \mathbf{W}_2]$.

Case IV: $N_a < N_e + N_b$ and $N_j \leq N_b$. For a start, let $\mathbf{V}_0 = \text{null}\{\mathbf{G}_1\}$ and $d_0 = (N_a - N_e)^+$. Secondly, denote $\mathbf{V}_0^c = \text{null}\{\mathbf{V}_0^H\} \in \mathbb{C}^{N_a \times N_e}$ and $\tilde{\mathbf{G}}_1 = \mathbf{G}_1 \mathbf{V}_0^c$. Invoking the GSVD Transform of $(\mathbf{H}_2^H, \tilde{\mathbf{G}}_1^H)$ yields

$$(\Psi_1, \Psi_2, \mathbf{D}_1, \mathbf{D}_2, \mathbf{X}, k_4, r_4, s_4) = \text{gsvd}(\mathbf{H}_2^H, \tilde{\mathbf{G}}_1^H). \quad (33)$$

Then let $\mathbf{W}_2 = \Psi_1(:, r_4 + 1 : r_4 + d_2)$ and $\mathbf{V}_2 = \mathbf{V}_0^c \Psi_2(:, c_4 + 1 : c_4 + d_4)$ in which $d_2 = \min\{s_4, \lfloor \frac{N_b - d_0}{2} \rfloor\}$ and $c_4 = r_4 + N_a - k_4$. Lastly, let $\hat{\mathbf{V}} = [\mathbf{V}_0, \mathbf{V}_2]$ and $\hat{\mathbf{W}} = \mathbf{W}_2$.

orthogonal with \mathbf{V}_2 by definition, thus

$$\begin{aligned} d^* &= \text{rank}\{[\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2]\} \\ &= \text{rank}\{[\mathbf{V}_0, \mathbf{V}_1]\} + \text{rank}\{\mathbf{V}_2\} \\ &= \text{rank}\{\mathbf{V}_0\} + \text{rank}\{\mathbf{V}_1\} + \text{rank}\{\mathbf{V}_2\} \\ &= \min\{d_0 + d_1 + d_2, N_a\}. \end{aligned}$$

In *Case IV*, $\hat{\mathbf{V}} = [\mathbf{V}_0, \mathbf{V}_2]$ and $\hat{\mathbf{W}} = \mathbf{W}_2$. According to (33), $\text{span}(\mathbf{H}_2 \mathbf{W}_2) = \text{span}(\mathbf{G}_1 \mathbf{V}_2)$, which, together with $\mathbf{G}_1 \mathbf{V}_0 = \mathbf{0}$, gives $\text{span}(\mathbf{H}_2 \hat{\mathbf{W}}) = \text{span}(\mathbf{G}_1 \hat{\mathbf{V}})$. In addition, $\text{span}(\mathbf{H}_1 \hat{\mathbf{V}}) \cap \text{span}(\mathbf{G}_2 \hat{\mathbf{W}}) = \{\mathbf{0}\}$ due to $d_2 = \min\{s_4, \lfloor \frac{N_b - d_0}{2} \rfloor\}$. Thus, $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ is feasible to (30). Furthermore, \mathbf{V}_0 is orthogonal with \mathbf{V}_2 by definition, therefore

$$\begin{aligned} d^* &= \text{rank}\{[\mathbf{V}_0, \mathbf{V}_2]\} \\ &= \text{rank}\{\mathbf{V}_0\} + \text{rank}\{\mathbf{V}_2\} \\ &= \min\{d_0 + d_2, N_a\}. \end{aligned}$$

Summarizing the above four cases, we can rewrite d^* into a more compact form as follows:

$$d^* = \min\{d_0^* + d_1^* + d_2^*, N_a, N_b\}, \quad (34)$$

TABLE II: Summary of the closed-form results on $d^*(s.d.o.f.)$

Inequalities on the number of antennas at terminals	$d^*(s.d.o.f.)$
$N_a \geq N_e + N_b$	$\min\{N_a, N_b\}$
$N_j \geq N_e + N_b$	
$2N_b + N_e - N_j \leq N_a < N_e + N_b$ $N_b < N_j < N_e + N_b$	
$N_b + N_e - N_j < N_a < 2N_b + N_e - N_j$ $N_b < N_j < N_e + N_b$	$N_a + N_j - (N_b + N_e) + \min\{s, \lfloor \frac{2N_b + N_e - N_a - N_j}{2} \rfloor\}$ $s = \min\{N_b + N_e - N_j, N_e\} + \min\{N_j, N_e\} - N_e$
$N_e < N_a < N_e + N_b, N_j \leq N_b$	$N_a - N_e + \min\{s, \lfloor \frac{N_b + N_e - N_a}{2} \rfloor\}, s = \min\{N_j, N_e\}$
$N_a \leq N_b + N_e - N_j, N_b < N_j < N_e + N_b$	$\min\{s, \lfloor \frac{N_b}{2} \rfloor\}$
$N_a \leq N_e, N_j \leq N_b$	$s = \min\{N_a, N_e\} + \min\{N_j, N_e\} - \min\{N_a + N_j, N_e\}$

in which

$$d_0^* = (N_a - N_e)^+ \quad (35a)$$

$$d_1^* = (\min\{N_a, N_e\} + (N_j - N_b)^+ - N_e)^+ \quad (35b)$$

$$d_2^* = \min\{s, (\lfloor \frac{N_b - (d_0^* + d_1^*)}{2} \rfloor)^+\}, \quad (35c)$$

where $s = \min\{N_a - (d_0^* + d_1^*), N_e\} + \min\{N_j, N_e\} - \min\{N_a - (d_0^* + d_1^*) + N_j, N_e\}$.

To gain more insight into d^* , we give Table II which clarifies the connection of d^* to the number of antennas at each terminal.

Lemma 5: On d defined in (30), we have $d = d^$ where d^* is given in (34).*

Proof: See Appendix F

Remark: According to Lemma 5, it is straight-forward that the feasible solution $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ given in Table I is also the optimal solution to (30).

Theorem 2: Consider a helper-assisted MIMO Gaussian wiretap channel, as depicted in Fig.1, where the source, the legitimate receiver, the eavesdropper and an external helper are equipped with N_a, N_b, N_e and N_j antennas, respectively. The maximal achievable secure degrees of freedom

$$s.d.o.f. = d^*, \quad (36)$$

where d^* is given in (34).

Proof: Combining Lemma 4 and Lemma 5, it is clear that $s.d.o.f. = d^*$. This completes the proof.

Corollary 1: The feasible point $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ for the optimization problem of (30), given in Table I, serves as a s.d.o.f.-optimal solution to the original SRM problem in (3). It achieves the maximal s.d.o.f.. Moreover, Table II clarifies the maximal achievable s.d.o.f. of a helper-assisted MIMO Gaussian wiretap channel, and reveals its specific connection to the number of antennas at each terminal.

Proof: With Theorem 2, it is straight-forward to arrive at these conclusions.

Corollary 2: When $N_b > 1$, the maximal achievable s.d.o.f. of a helper-assisted MIMO Gaussian wiretap channel is zero if and only if $N_e \geq N_a + N_j$. When $N_b = 1$, the maximal achievable s.d.o.f. of a helper-assisted MIMO Gaussian wiretap channel is zero if and only if $N_e \geq N_a + N_j - 1$.

Proof: See Appendix G

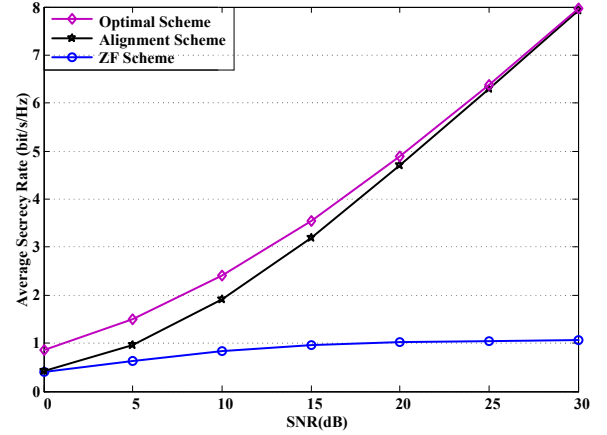


Fig. 2: Average secrecy rate versus SNR, $N_a = N_e = 3$, $N_j = 2$, $N_b = 1$

V. NUMERICAL RESULTS

In this section, we give numerical results to show the secrecy rate performance of the proposed schemes and validate our theoretical findings. All channels are assumed to be quasi-static flat Rayleigh fading and independent of each other, with entries distributed as $\mathcal{CN}(0, 1)$. The noise vector at each receiver is assumed to be AWGN, with i.i.d. entries distributed as $\mathcal{CN}(0, 1)$. In each figure, details on the number of antennas at each terminal will be depicted. Results are averaged over 1000 independent channel trials.

We first test the secrecy rate performance of our proposed schemes and compare them with the existing method.

Fig.2 illustrates the secrecy rate performance of the single-antenna legitimate receiver case. The lines labeled as “Optimal Scheme” and “Alignment Scheme” illustrate the secrecy rate performance of C_s and C_s^{sub} in (8) and (20), respectively. The line labeled as “ZF Scheme” [10] gives the secrecy rate achieved by the scheme which completely nulls out the jamming signal at the legitimate receiver and matches the message signal with the source-receiver (legitimate) channel. It shows that both C_s and C_s^{sub} increase linearly with SNR. In contrast, there exists a performance ceiling on the secrecy rate achieved by the ZF Scheme.

Fig. 3 illustrates the secrecy rate performance of the multi-antenna legitimate receiver case. The bar labeled as “Align-

ment Scheme” shows the secrecy rate result of our proposed heuristic method. In such case, closed-form precoding matrices $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ are given in Table I and the total power is equally distributed over all message signal streams and jamming signal streams. The bar labeled as “Gauss-Seidel Approach” shows the secrecy rate performance of our proposed iterative algorithm in Section IV. A. As stated in Corollary 1, $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ given in Table I, actually acts as a s.d.o.f.-optimal solution to the original SRM problem in (3). So the initial point is set as the closed-form solutions in the Alignment Scheme. For comparison, Fig. 3 also plots the secrecy rate performance of the method proposed in [4], wherein the secrecy rate maximization method is derived under a power covariance constraint. Thus, to find the maximal achievable secrecy rate under an average power constraint, we have to solve [5, equation (41)] [29]

$$R_s(P) = \max_{\mathbf{S} \succeq \mathbf{0}, \text{tr}\{\mathbf{S}\} \leq P} R_s(\mathbf{S}).$$

That is, numerical search over the power covariance matrix \mathbf{S} is performed to compute $R_s(P)$. Since such numerical search is based on random choices of \mathbf{S} , it is difficult to decide when to stop it. To deal with this issue, we first determine the run time of our Gauss-Seidel based algorithm of Section IV. A, which terminates when the relative secrecy rate improvement between two adjacent iterations is less than 10^{-2} . We then run the algorithm proposed in [4] for the same run time. It is worthwhile to note that our proposed Alignment Scheme has closed-form solutions, so it is the most computationally inexpensive scheme. Fig. 3 shows that, with the same run time, our proposed Gauss-Seidel based algorithm achieves higher secrecy rate than the algorithm proposed in [4]. More encouragingly, it can be seen that the Alignment Scheme achieves nearly the same secrecy rate as the Gauss-Seidel Approach based algorithm in the high SNR regime. This is consistent with the fact that the Alignment Scheme is s.d.o.f.-optimal, thus near-optimal at high SNR. Besides, it can be seen that the Alignment Scheme achieves higher secrecy rate than the algorithm proposed in [4] in the high and medium SNR regimes. To gain more insight into the Gauss-Seidel based algorithm, Fig. 4 plots the convergence of it. Results show that our proposed Gauss-Seidel Approach converges very fast and stabilizes after several loops.

We then test the achievable s.d.o.f. performance for the helper-assisted MIMO wiretap channel and validate the theoretical results of Section IV. B.

In Fig. 5, the stem labeled as “Theoretical Results” shows the theoretical maximal achievable s.d.o.f. according to Table II. The stem labeled as “Numerical Results” shows the s.d.o.f. achieved by the proposed Alignment Scheme. In the proposed Alignment Scheme, closed-form precoding matrices $\{\hat{\mathbf{V}}, \hat{\mathbf{W}}\}$ are given in Table I and the total power is equally distributed over all message signal streams and jamming signal streams. The total power P is set as 50dB. For each channel trial, we substitute the closed-form solution into (3) and compute the secrecy rate C_s^o . We then compute the s.d.o.f. as the rate at which the secrecy rate C_s^o scales with $\log P$, i.e., $C_s^o / \log P$. It can be seen that the theoretical results almost coincide with

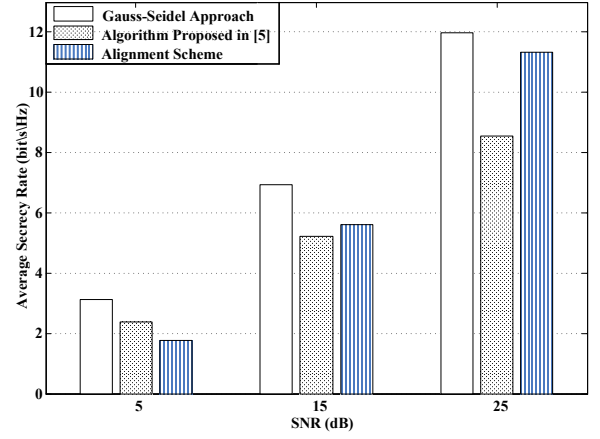


Fig. 3: Average secrecy rate versus SNR, $N_a = N_b = 3$, $N_j = 4$, $N_e = 3$

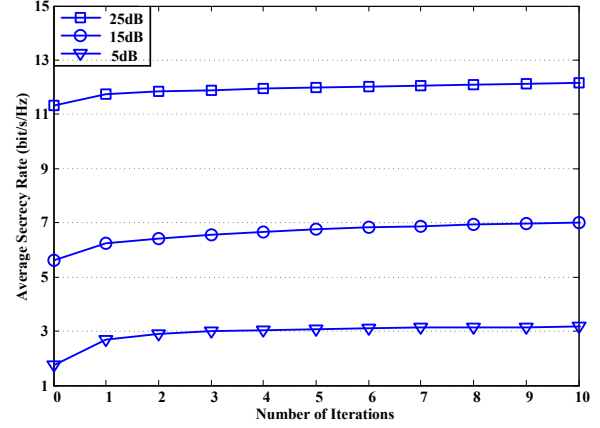


Fig. 4: Average secrecy rate versus the number of iterations, $N_j = 4$, $N_a = N_b = 3$, $N_e = 3$

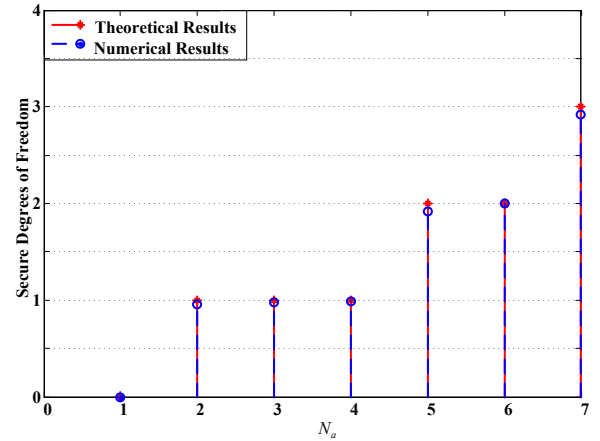


Fig. 5: *s.d.o.f.* versus N_a , $N_j = 3$, $N_b = 3$, $N_e = 4$

the numerical results.

Fig. 6 and Fig. 7 plot the maximal achievable s.d.o.f. for the helper-assisted MIMO Gaussian wiretap channel under various antenna configurations, according to Table II. Results show that for the case of single-antenna legitimate receiver, the maximal achievable s.d.o.f. is zero if and only if $N_e \geq N_a + N_j - 1$, while for the case of multi-antenna legitimate receiver, the maximal achievable s.d.o.f. is zero if and only if

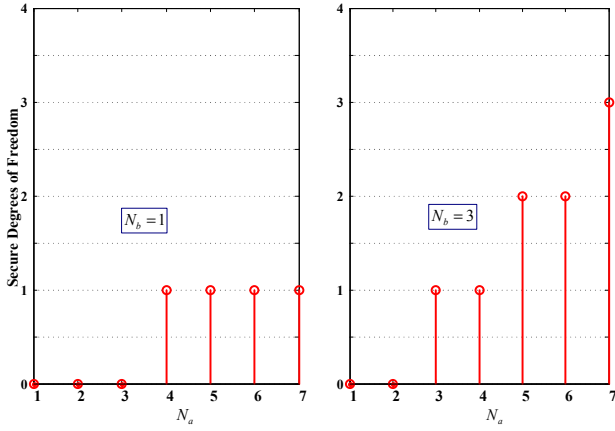


Fig. 6: *s.d.o.f.* versus N_a , $N_j = 2$, $N_e = 4$

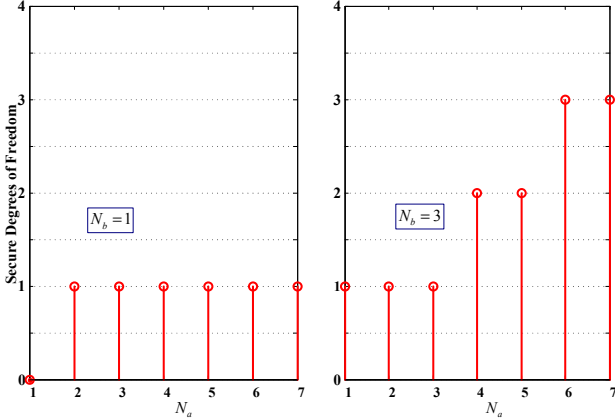


Fig. 7: *s.d.o.f.* versus N_a , $N_j = 4$, $N_e = 4$

$N_e \geq N_a + N_j$. Further, it is illustrated that the maximal achievable s.d.o.f. benefits from the increasing number of antennas at any of the three terminals, i.e., the source, the legitimate receiver and the external helper. These results are consistent with the theoretical findings of Section IV. B.

VI. CONCLUSION

We have studied the secrecy capacity of the MIMO Gaussian wiretap channel, where a multi-antenna external helper is available. For the special case of single-antenna legitimate receiver, we have obtained the secrecy capacity using a combination of convex optimization and one-dimensional search. For the case of multi-antenna legitimate receiver, we have reformulated the original nonconvex SRM problem into several convex subproblems. By doing so, we have been able to provide an iterative algorithm, which attains a fairly good secrecy rate performance. In addition, we have addressed the s.d.o.f. maximization analytically. Specifically, we have obtained an analytical s.d.o.f.-optimal solution to the original SRM problem, based on which, we have obtained the maximal achievable s.d.o.f. in closed-form. These results uncovered the connection between the maximal achievable s.d.o.f. and the system parameters, thus shedding light on how the secrecy capacity of a helper-assisted MIMO Gaussian wiretap channel behaves. Numerical results have validated the theoretical findings and confirmed the efficacy of our proposed schemes.

APPENDIX A PROOF OF PROPOSITION 1

The associated Lagrangian of (13) is

$$\begin{aligned} \mathcal{L} = & \text{tr}\{\mathbf{Q}_a + \mathbf{Q}_j\} + \mu[f(\tau)(1 + \mathbf{g}_2\mathbf{Q}_j\mathbf{g}_2^H) - \mathbf{h}_1\mathbf{Q}_a\mathbf{h}_1^H] \\ & + \text{tr}\{\mathbf{Z}_1[\mathbf{G}_1\mathbf{Q}_a\mathbf{G}_1^H - \tau(\mathbf{I} + \mathbf{H}_2\mathbf{Q}_j\mathbf{H}_2^H)]\} \\ & - \text{tr}\{\mathbf{Z}_a\mathbf{Q}_a\} - \text{tr}\{\mathbf{Z}_j\mathbf{Q}_j\}, \end{aligned} \quad (37)$$

where \mathbf{Z}_1 , \mathbf{Z}_a , \mathbf{Z}_j and μ are dual variables associated with the inequalities in (13). The optimization problem of (13) is convex with part of the Karush-Kuhn-Tucker (KKT) conditions as follows:

$$\mathbf{Z}_a = \mathbf{I} - \mu\mathbf{h}_1^H\mathbf{h}_1 + \mathbf{G}_1^H\mathbf{Z}_1\mathbf{G}_1 \quad (38a)$$

$$\mathbf{Z}_a\mathbf{Q}_a = \mathbf{0} \quad (38b)$$

$$\mathbf{Z}_1 \succeq \mathbf{0}, \mathbf{Z}_a \succeq \mathbf{0}, \mu \geq 0. \quad (38c)$$

Substituting (38a) into (38b), we arrive at

$$(\mathbf{I} + \mathbf{G}_1^H\mathbf{Z}_1\mathbf{G}_1)\mathbf{Q}_a = \mu\mathbf{h}_1^H\mathbf{h}_1\mathbf{Q}_a. \quad (39)$$

Since $\mathbf{I} + \mathbf{G}_1^H\mathbf{Z}_1\mathbf{G}_1 \succ \mathbf{0}$, so $\text{rank}\{\mathbf{Q}_a\} = \text{rank}\{(\mathbf{I} + \mathbf{G}_1^H\mathbf{Z}_1\mathbf{G}_1)\mathbf{Q}_a\} = \text{rank}\{\mu\mathbf{h}_1^H\mathbf{h}_1\mathbf{Q}_a\} \leq 1$. In addition, $\text{rank}\{\mathbf{Q}_a\} = 0$ implies that $\mathbf{Q}_a = \mathbf{0}$, which contradicts the positive secrecy rate requirement. Thus, $\text{rank}\{\mathbf{Q}_a\} = 1$. This completes the proof.

APPENDIX B PROOF OF PROPOSITION 2

For notational simplicity, let

$$\Phi(\mathbf{Q}_a, \mathbf{Q}_j) = \mathbf{h}_1\mathbf{Q}_a\mathbf{h}_1^H / (1 + \mathbf{g}_2\mathbf{Q}_j\mathbf{g}_2^H).$$

Denote the optimal solution to (11) as $\{\bar{\mathbf{Q}}_a, \bar{\mathbf{Q}}_j\}$. Apparently, the point $\{\mathbf{Q}_a, \bar{\mathbf{Q}}_j\}$ is also feasible to (13). Thus we have

$$\text{tr}\{\hat{\mathbf{Q}}_a + \hat{\mathbf{Q}}_j\} \leq \text{tr}\{\bar{\mathbf{Q}}_a + \bar{\mathbf{Q}}_j\} \leq P,$$

which indicates the point $\{\hat{\mathbf{Q}}_a, \hat{\mathbf{Q}}_j\}$ is feasible to (11). According to the definition of $f(\tau)$, we see $\Phi(\hat{\mathbf{Q}}_a, \hat{\mathbf{Q}}_j) \leq f(\tau)$. Moreover, from (13), $\Phi(\hat{\mathbf{Q}}_a, \hat{\mathbf{Q}}_j) \geq f(\tau)$. Thus,

$$\Phi(\hat{\mathbf{Q}}_a, \hat{\mathbf{Q}}_j) = f(\tau),$$

which implies that the point $\{\hat{\mathbf{Q}}_a, \hat{\mathbf{Q}}_j\}$ is also the optimal solution to (11). This completes the proof.

APPENDIX C PROOF OF THE EQUALITIES IN (16)

Apparently, by the definition of (15a), (16a) holds true.

According to [24], for any given matrix of \mathbf{A} , it holds that

$$\text{span}(\mathbf{A}^H) = \text{null}(\mathbf{A})^\perp. \quad (40)$$

Applying (40) to (15b), we get

$$\begin{aligned} \text{span}(\mathbf{H})^\perp \cap \text{span}(\mathbf{G}) &= \text{null}(\mathbf{H}^H) \cap \text{null}(\mathbf{G}^H)^\perp \\ &= \text{null}(\mathbf{H}^H) / [\text{null}(\mathbf{H}^H) \cap \text{null}(\mathbf{G}^H)] \\ &= \text{null}(\mathbf{H}^H) / \text{null}([\mathbf{H}^H, \mathbf{G}^H]^T). \end{aligned} \quad (41)$$

In addition, $\text{null}([\mathbf{H}^H, \mathbf{G}^H]^T) \subset \text{null}(\mathbf{H}^H)$ by definition, so we have $p = \dim\{\text{null}(\mathbf{H}^H)\} - \dim\{\text{null}([\mathbf{H}^H, \mathbf{G}^H]^T)\} = N - \min\{M, N\} - (N - k) = k - \min\{M, N\}$.

Similarly, we can prove (16b)-(16d). This completes the proof.

APPENDIX D

PROOF OF THE LOWER BOUND ON C_s IN (24)

The optimization problem of (22) can be solved by resorting to carefully mathematical deductions. Let $y = 1 + cP + (b - c)x$, $\alpha = \min\{b, c\}$ and $\beta = \max(b, c)$. Then, $y \in [1 + \alpha P, 1 + \beta P]$. Substituting $x = (y - (1 + cP))/(b - c)$ into (23) and making some mathematical transformations yield

$$\eta(y) = \kappa - (Ay + B/y), \quad (42)$$

$$\text{in which } \kappa = \frac{a(1 + cP) + c - b}{c - b} + \frac{[2a(1 + cP) + c - b]b}{(c - b)^2},$$

$$A = \frac{ac}{(c - b)^2} \text{ and } B = \frac{b(1 + cP)[a(1 + cP) + c - b]}{(c - b)^2}.$$

Resorting to (42), the optimization problem of (22) can be transformed into a new optimization problem that searches for y as follows:

$$\eta_{\max} \triangleq \max_{1 + \alpha P \leq y \leq 1 + \beta P} \kappa - (Ay + B/y). \quad (43)$$

- 1) For the case of $a(1 + cP) > b - c$, $B > 0$. Thus, $\eta(y)$ is increasing in y when $y < y_0$, and decreasing in y when $y > y_0$. Herein, $y_0 = \sqrt{B/A} = \sqrt{b(1 + cP)[a(1 + cP) - b + c]/(ac)}$. Therefore, if $1 + \alpha P \leq y_0 \leq 1 + \beta P$, $\eta_{\max} = \kappa - 2\sqrt{AB}$. Otherwise, $\eta(y)$ achieves its maximal value at the two endpoints ($y = 1 + \alpha P$ or $y = 1 + \beta P$), and $\eta_{\max} = \max\{(1 + aP)/(1 + bP), 1\}$.
- 2) For the case of $a(1 + cP) \leq b - c$, $B \leq 0$. Thus, $\eta(y)$ decreases monotonically with respect to y . It achieves its maximal value at the two endpoints, and $\eta_{\max} = \max\{(1 + aP)/(1 + bP), 1\}$.

Summarizing, if $1 + \alpha P \leq y_0 \leq 1 + \beta P$, then $\eta_{\max} = \kappa - 2\sqrt{AB}$ and

$$C_s^{\text{sub}} = \log\left(\frac{c(1 + aP)[c + b - 2\sqrt{bc}\sqrt{1 + (c - b)/(a + acP)}]}{(c - b)^2} + \frac{c(a - b) + 2(c - a)\sqrt{bc}\sqrt{1 + (c - b)/(a + acP)}}{(c - b)^2}\right), \quad (44)$$

where the optimal solution

$$x^* = (y_0 - (1 + cP))/(b - c) = \frac{\sqrt{b(1 + cP)[a(1 + cP) - b + c]/(ac)} - (1 + cP)}{b - c}. \quad (45)$$

Otherwise, when $a > b$, $\eta_{\max} = (1 + aP)/(1 + bP)$ and $C_s^{\text{sub}} = \log(1 + aP)/(1 + bP)$, where the optimal solution $x^* = P$; when $a \leq b$, $\eta_{\max} = 1$ and $C_s^{\text{sub}} = 0$, where the optimal solution $x^* = 0$. In the sequel, we refer to these two solutions $x^* = 0$ and $x^* = P$ as the trivial solutions.

Consider the nontrivial solution of (45). When the total transmit power P is big enough,

$$C_s^{\text{sub}} \approx \log(aP) - 2\log(1 + \sqrt{b/c}). \quad (46)$$

This completes the proof.

APPENDIX E

PROOF OF LEMMA 4

Before proceeding, we first give two critical properties on matrix that will be used in the following analyses. That is, for any given matrices \mathbf{A} and \mathbf{B} , if \mathbf{B} is invertible, then

$$\text{span}(\mathbf{A}) = \text{span}(\mathbf{AB}) \quad (47)$$

$$\text{rank}\{\mathbf{A}\} = \text{rank}\{\mathbf{AB}\}. \quad (48)$$

Firstly, $\text{span}(\mathbf{A}) = \text{span}(\mathbf{ABB}^{-1}) \subset \text{span}(\mathbf{AB})$. Secondly, $\text{span}(\mathbf{A}) \supset \text{span}(\mathbf{AB})$. Therefore, the equality (47) holds true. With (47), it is clear that the equality (48) holds true.

Given an arbitrary point of $\{\mathbf{V}, \mathbf{W}\}$, with the definition in (4), we can re-express the achieved s.d.o.f. as follows:

$$h(\mathbf{V}, \mathbf{W}) = \text{rank}\{\mathbf{H}_1 \mathbf{V}\} - m(\mathbf{V}, \mathbf{W}) - n(\mathbf{V}, \mathbf{W}), \quad (49)$$

in which $m(\mathbf{V}, \mathbf{W}) = \dim\{\text{span}(\mathbf{G}_1 \mathbf{V})/\text{span}(\mathbf{H}_2 \mathbf{W})\}$ and $n(\mathbf{V}, \mathbf{W}) = \dim\{\text{span}(\mathbf{G}_2 \mathbf{W}) \cap \text{span}(\mathbf{H}_1 \mathbf{V})\}$.

Assume that $\{\bar{\mathbf{V}}, \bar{\mathbf{W}}\}$ is the optimal solution to (30), then we have $\text{span}(\mathbf{G}_1 \bar{\mathbf{V}}) \subset \text{span}(\mathbf{H}_2 \bar{\mathbf{W}})$ and $\text{span}(\mathbf{G}_2 \bar{\mathbf{W}}) \cap \text{span}(\mathbf{H}_1 \bar{\mathbf{V}}) = \{\mathbf{0}\}$. The achieved s.d.o.f. $h(\bar{\mathbf{V}}, \bar{\mathbf{W}}) = \text{rank}\{\mathbf{H}_1 \bar{\mathbf{V}}\} = d$. In addition, $s.d.o.f. \geq h(\bar{\mathbf{V}}, \bar{\mathbf{W}})$ by definition. Thus, $s.d.o.f. \geq d$. As such, to complete the proof of Lemma 4, we only need to prove $s.d.o.f. \leq d$. In the sequel, we show that, for any given point of $\{\mathbf{V}, \mathbf{W}\}$, we can always find another feasible point for the problem of (30), $\{\mathbf{V}', \mathbf{W}'\}$, such that $h(\mathbf{V}, \mathbf{W}) \leq \text{rank}\{\mathbf{H}_1 \mathbf{V}'\} \leq d$, thus giving the proof of $s.d.o.f. \leq d$.

Without loss of generality, denote $\mathbf{V} \in \mathbb{C}^{N_a \times K_a}$ and $\mathbf{W} \in \mathbb{C}^{N_j \times K_j}$. With the GSVD Transform of $(\mathbf{W}^H \mathbf{G}_2^H, \mathbf{V}^H \mathbf{H}_1^H)$, we obtain unitary matrices $\hat{\Psi}_1 \in \mathbb{C}^{K_j \times K_j}$ and $\hat{\Psi}_2 \in \mathbb{C}^{K_a \times K_a}$, non-negative diagonal matrices $\hat{\mathbf{D}}_1 \in \mathbb{C}^{K_j \times k_5}$ and $\hat{\mathbf{D}}_2 \in \mathbb{C}^{K_a \times k_5}$, and a matrix $\hat{\mathbf{X}} \in \mathbb{C}^{N_r \times k_5}$ with $\text{rank}\{\hat{\mathbf{X}}\} = k_5$, such that

$$\mathbf{G}_2 \mathbf{W} \hat{\Psi}_1 = \hat{\mathbf{X}} \hat{\mathbf{D}}_1^H \quad (50a)$$

$$\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2 = \hat{\mathbf{X}} \hat{\mathbf{D}}_2^H, \quad (50b)$$

where $k_5 = \min\{K_a + K_j, N_r\}$, $r_5 = k_5 - \min\{K_a, N_r\}$ and $s_5 = \min\{K_a, N_r\} + \min\{K_j, N_r\} - k_5$.

Let

$$\hat{\Psi}_1^1 = \hat{\Psi}_1(:, r_5 + 1 : r_5 + s_5) \quad (51a)$$

$$\hat{\Psi}_1^0 = [\hat{\Psi}_1(:, 1 : r_5), \hat{\Psi}_1(:, r_5 + s_5 : K_j)] \quad (51b)$$

$$\hat{\Psi}_2^1 = \hat{\Psi}_2(:, c_{in} + 1 : c_{in} + s_5) \quad (51c)$$

$$\hat{\Psi}_2^0 = [\hat{\Psi}_2(:, 1 : c_{in}), \hat{\Psi}_2(:, c_{in} + s_5 + 1 : K_a)], \quad (51d)$$

in which $c_{in} = r_5 + K_a - k_5$. Since $\hat{\Psi}_1$ and $\hat{\Psi}_2$ are invertible matrices, $\hat{\Psi}_1^0 = [\hat{\Psi}_1^0, \hat{\Psi}_1^1]$ and $\hat{\Psi}_2^0 = [\hat{\Psi}_2^0, \hat{\Psi}_2^1]$ are also invertible matrices. Applying (47) and (48), we have

$$h(\mathbf{V}, \mathbf{W}) = h(\mathbf{V} \hat{\Psi}_2^1, \mathbf{W} \hat{\Psi}_1^1) \quad (52a)$$

$$= \text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0\} - m(\mathbf{V} \hat{\Psi}_2^1, \mathbf{W} \hat{\Psi}_1^1) \quad (52b)$$

$$\leq \text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0\} - m(\mathbf{V} \hat{\Psi}_2^0, \mathbf{W} \hat{\Psi}_1^1), \quad (52c)$$

in which (52b) can be justified with $\text{span}(\mathbf{G}_2 \mathbf{W} \hat{\Psi}_1^1) \cap \text{span}(\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^1) = \text{span}(\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^1)$. Besides, (52c) comes from the fact that $m(\mathbf{V} \hat{\Psi}_2^1, \mathbf{W} \hat{\Psi}_1^1) \geq m(\mathbf{V} \hat{\Psi}_2^0, \mathbf{W} \hat{\Psi}_1^1)$.

With the *GSVD Transform* of $((\mathbf{H}_2 \mathbf{W} \hat{\Psi}_1')^H, (\mathbf{G}_1 \mathbf{V} \hat{\Psi}_2^0)^H)$, we obtain unitary matrices $\check{\Psi}_1 \in \mathbb{C}^{K_j \times K_j}$ and $\check{\Psi}_2 \in \mathbb{C}^{(K_a - s_5) \times (K_a - s_5)}$, non-negative diagonal matrices $\check{\mathbf{D}}_1 \in \mathbb{C}^{K_j \times k_6}$ and $\check{\mathbf{D}}_2 \in \mathbb{C}^{(K_a - s_5) \times k_6}$, and a matrix $\check{\mathbf{X}} \in \mathbb{C}^{N_e \times k_6}$ with $\text{rank}\{\check{\mathbf{X}}\} = k_6$, such that

$$\mathbf{H}_2 \mathbf{W} \hat{\Psi}_1' \check{\Psi}_1 = \check{\mathbf{X}} \check{\mathbf{D}}_1^H \quad (53a)$$

$$\mathbf{G}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2 = \check{\mathbf{X}} \check{\mathbf{D}}_2^H, \quad (53b)$$

where $k_6 = \min\{K_a - s_5 + K_j, N_e\}$, $r_6 = k_6 - \min\{K_a - s_5, N_e\}$ and $s_6 = \min\{K_j, N_e\} + \min\{K_a - s_5, N_e\} - k_6$.

Let

$$\check{\Psi}_1^1 = \check{\Psi}_1(:, 1 : r_6 + s_6) \quad (54a)$$

$$\check{\Psi}_1^0 = \check{\Psi}_1(:, r_6 + s_6 : K_j) \quad (54b)$$

$$\check{\Psi}_2^1 = \check{\Psi}_2(:, 1 : \check{c}_{\text{in}} + s_6) \quad (54c)$$

$$\check{\Psi}_2^0 = \check{\Psi}_2(:, \check{c}_{\text{in}} + s_6 + 1 : K_a - s_5), \quad (54d)$$

in which $\check{c}_{\text{in}} = r_6 + (K_a - s_5) - k_6$. Since $\check{\Psi}_1$ and $\check{\Psi}_2$ are invertible matrices, so $\check{\Psi}_1' = [\check{\Psi}_1^0, \check{\Psi}_1^1]$ and $\check{\Psi}_2' = [\check{\Psi}_2^0, \check{\Psi}_2^1]$ are also invertible matrices. Applying (47) and (48), we have

$$\begin{aligned} & \text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0\} - m(\mathbf{V} \hat{\Psi}_2^0, \mathbf{W} \hat{\Psi}_1') \\ &= \text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2'\} - m(\mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2', \mathbf{W} \hat{\Psi}_1' \check{\Psi}_1') \end{aligned} \quad (55a)$$

$$= \text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2'\} - \text{rank}\{\check{\Psi}_2^0\} \quad (55b)$$

$$\leq \text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^1\}, \quad (55c)$$

where due to $\text{span}(\mathbf{G}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2') / \text{span}(\mathbf{H}_2 \mathbf{W} \hat{\Psi}_1' \check{\Psi}_1') = \text{span}(\mathbf{G}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^1)$, (55b) holds true. Besides, (55c) holds true due to $\text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2'\} \leq \text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^1\} + \text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^0\}$ and $\text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^0\} \leq \text{rank}\{\check{\Psi}_2^0\}$.

Combining (52) with (55), we arrive at

$$h(\mathbf{V}, \mathbf{W}) \leq \text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^1\}. \quad (56)$$

With (53) and (54), we arrive at $m(\mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^1, \mathbf{W} \hat{\Psi}_1' \check{\Psi}_1') = 0$, thus

$$\text{span}(\mathbf{G}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^1) \subset \text{span}(\mathbf{H}_2 \mathbf{W} \hat{\Psi}_1' \check{\Psi}_1'). \quad (57)$$

In addition, with (50) and (51), we get $n(\mathbf{V} \hat{\Psi}_2^0, \mathbf{W} \hat{\Psi}_1') = 0$. So $\text{span}(\mathbf{G}_2 \mathbf{W} \hat{\Psi}_1') \cap \text{span}(\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0) = \{0\}$, which, together with the facts that $\text{span}(\mathbf{G}_2 \mathbf{W} \hat{\Psi}_1') = \text{span}(\mathbf{G}_2 \mathbf{W} \hat{\Psi}_1' \check{\Psi}_1')$ and $\text{span}(\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0) \supset \text{span}(\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^1)$, gives

$$\text{span}(\mathbf{G}_2 \mathbf{W} \hat{\Psi}_1' \check{\Psi}_1') \cap \text{span}(\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^1) = \{0\}. \quad (58)$$

Combining (57) with (58), we know $\{\mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^1, \mathbf{W} \hat{\Psi}_1' \check{\Psi}_1'\}$ is a feasible point for the problem of (30). By definition, $\text{rank}\{\mathbf{H}_1 \mathbf{V} \hat{\Psi}_2^0 \check{\Psi}_2^1\} \leq d$, which, together with (56), indicates that $h(\mathbf{V}, \mathbf{W}) \leq d$.

Because the above derivations hold true for any given point of $\{\mathbf{V}, \mathbf{W}\}$, we conclude that $s.d.o.f. \leq d$. This completes the proof.

APPENDIX F PROOF OF LEMMA 5

Clearly, $d \geq d^*$ holds true. So if we can further prove $d \leq d^*$, then the proof of Lemma 5 is completed. In the

following text, we give the proof of $d \leq d^*$ by contradiction. Assume that there exists a feasible point $\{\tilde{\mathbf{V}}, \tilde{\mathbf{W}}\}$ of (30), where $\tilde{\mathbf{V}} \in \mathbb{C}^{N_a \times K_a}$, $d^\# \triangleq \text{rank}\{\mathbf{H}_1 \tilde{\mathbf{V}}\}$ and $K_a = d^\# > d^*$. In such case, we have $\text{rank}\{\tilde{\mathbf{V}}\} = d^\#$ due to $d^\# = \text{rank}\{\mathbf{H}_1 \tilde{\mathbf{V}}\} \leq \text{rank}\{\tilde{\mathbf{V}}\} \leq K_a = d^\#$. Besides, by definition, $d^\# \leq \min\{N_a, N_b\}$ always holds true. In the sequel, we discuss the four cases in Table I and give contradictions one by one.

In *Case I* and *Case II*, $d^* = \min\{N_a, N_b\}$. The assumption $d^\# > d^*$ implies $d^\# > \min\{N_a, N_b\}$, which contradicts the fact $d^\# \leq \min\{N_a, N_b\}$.

In *Case III*, when $d^* = \min\{N_a, N_b\}$, the assumption $d^\# > d^* = N_b$ contradicts the fact $d^\# \leq \min\{N_a, N_b\}$. As such, we only need to focus on the case of $d^* = d_0 + d_1 + d_2$, where $d_0 = (N_a - N_e)^+$, $d_1 = s_3$ and $d_2 = \min\{s_4, \lfloor \frac{N_b - (d_0 + d_1)}{2} \rfloor\}$.

- 1) For the case of $s_4 \leq \lfloor \frac{N_b - (d_0 + d_1)}{2} \rfloor$, $d^* = d_0 + d_1 + s_4$. In addition, $d^\# > d^*$. So $d^\# > d_0 + d_1 + s_4$. Therefore $d^\# - d_0 > s_3 + s_4$, which contradicts (30b). The explanation is as follows. According to (32), $s_4 = \min\{N_j, N_e\} + \min\{N_a - d_0 - d_1, N_e\} - \min\{N_j + N_a - d_0 - d_1, N_e\}$. With the *GSVD Transform* of $(\mathbf{H}_2^H, \mathbf{G}_1^H)$, $s = \dim\{\text{span}(\mathbf{H}_2) \cap \text{span}(\mathbf{G}_1)\} = \min\{N_j, N_e\} + \min\{N_a, N_e\} - \min\{N_j + N_a, N_e\}$. It is easy to verify that $s \leq s_3 + s_4$. In addition, to satisfy (30b) we should have $\text{rank}\{\mathbf{G}_1 \tilde{\mathbf{V}}\} \leq s$. Thus $\text{rank}\{\mathbf{G}_1 \tilde{\mathbf{V}}\} \leq s_3 + s_4$. Moreover, $\text{rank}\{\tilde{\mathbf{V}}\} - d_0 \leq \min\{N_a, N_e\}$ due to the fact $\text{rank}\{\tilde{\mathbf{V}}\} \leq N_a$, so $\text{rank}\{\mathbf{G}_1 \tilde{\mathbf{V}}\} = \text{rank}\{\tilde{\mathbf{V}}\} - d_0 = d^\# - d_0$. Therefore, $d^\# - d_0 \leq s_3 + s_4$, which gives the contradiction.
- 2) For the case of $s_4 > \lfloor \frac{N_b - (d_0 + d_1)}{2} \rfloor$, $d^* = d_0 + d_1 + \lfloor \frac{N_b - (d_0 + d_1)}{2} \rfloor$, which, together with the assumption $d^\# > d^*$, gives

$$d^\# > d_0 + d_1 + \lfloor \frac{N_b - (d_0 + d_1)}{2} \rfloor. \quad (59)$$

If $N_b - (d_0 + d_1)$ is an even number, (59) is equivalent to $2d^\# > N_b + d_0 + d_1$. Otherwise, $N_b - (d_0 + d_1)$ is an odd number, so (59) is equivalent to $2d^\# > N_b + d_0 + d_1 - 1$. In addition, $N_b + d_0 + d_1$ owns the same parity as $N_b - d_0 - d_1$, thus $N_b + d_0 + d_1 - 1$ is an even number. Therefore $2d^\# > N_b + d_0 + d_1$. To sum up, (59) indicates $2d^\# > N_b + d_0 + d_1$. Thus $d^\# - d_0 > N_b - d^\# + d_1$. Moreover, to satisfy (30c), we should have $N_b - d^\# + d_1 \geq \text{rank}\{\tilde{\mathbf{W}}\}$. So $d^\# - d_0 > \text{rank}\{\tilde{\mathbf{W}}\}$. However, $\text{rank}\{\mathbf{G}_1 \tilde{\mathbf{V}}\} = \text{rank}\{\tilde{\mathbf{V}}\} - d_0$ due to $\text{rank}\{\tilde{\mathbf{V}}\} - d_0 \leq \min\{N_a, N_e\}$. Thus, $\text{rank}\{\mathbf{G}_1 \tilde{\mathbf{V}}\} = d^\# - d_0 > \text{rank}\{\tilde{\mathbf{W}}\} \geq \text{rank}\{\mathbf{H}_2 \tilde{\mathbf{W}}\}$, which contradicts (30b).

In *Case IV*, $d^* = d_0 + d_2$, where $d_0 = (N_a - N_e)^+$ and $d_2 = \min\{s_4, \lfloor \frac{N_b - d_0}{2} \rfloor\}$. Since the analysis is similar to *Case III*, so in the sequel we only give the skeleton on it.

- 1) For the case of $s_4 \leq \lfloor \frac{N_b - d_0}{2} \rfloor$, $d^* = d_0 + s_4$, which, combined with the assumption $d^\# > d^*$, gives $d^\# > d_0 + s_4$. Thus $d^\# - d_0 > s_4$. However, with the *GSVD Transform* of $(\mathbf{H}_2^H, \mathbf{G}_1^H)$, $s = \dim\{\text{span}(\mathbf{H}_2) \cap \text{span}(\mathbf{G}_1)\} = s_4$. To satisfy (30b), we should have $\text{rank}\{\mathbf{G}_1 \tilde{\mathbf{V}}\} \leq s = s_4$, which, together with the fact $\text{rank}\{\tilde{\mathbf{V}}\} - d_0 \leq \min\{N_a, N_e\}$, indicates $d^\# - d_0 \leq s_4$.

2) For the case of $s_4 > \lfloor \frac{N_b - d_0}{2} \rfloor$, $d^* = d_0 + \lfloor \frac{N_b - d_0}{2} \rfloor$, which, together with the assumption $d^\# > d^*$, gives $2d^\# > N_b + d_0$. Thus $d^\# - d_0 > N_b - d^\#$. However, to satisfy (30c), we should have $N_b - d^\# \geq \text{rank}\{\tilde{\mathbf{W}}\}$. Therefore, $\text{rank}\{\tilde{\mathbf{V}}\} - d_0 = d^\# - d_0 > \text{rank}\{\tilde{\mathbf{W}}\}$, which, together with the fact $\text{rank}\{\mathbf{G}_1 \tilde{\mathbf{V}}\} = \text{rank}\{\tilde{\mathbf{V}}\} - d_0$, contradicts (30b).

Summarizing the above four cases, for any feasible points for the problem of (30), denoted by $\{\tilde{\mathbf{V}}, \tilde{\mathbf{W}}\}$ and $\tilde{\mathbf{V}} \in \mathbb{C}^{N_a \times K_a}$, if $K_a = d^\#$, $d^\# \leq d^*$. On the other hand, if $K_a > d^\#$, resorting to the singular value decomposition (SVD) of $\mathbf{H}_1 \tilde{\mathbf{V}}$, we can always find another feasible point $\{\tilde{\mathbf{V}}', \tilde{\mathbf{W}}'\}$ for the problem of (30), such that $\tilde{\mathbf{V}}' \in \mathbb{C}^{N_a \times K'_a}$ and $K'_a = \text{rank}\{\tilde{\mathbf{V}}'\} = \text{rank}\{\mathbf{H}_1 \tilde{\mathbf{V}}'\} = d^\#$. As such, the assumption $d^\# > d^*$ also contradicts the feasibility conditions in (30). So $d^\# \leq d^*$.

In conclusion, for any feasible points for the problem of (30), we should have $d^\# \leq d^*$. By definition, we arrive at $d \leq d^*$. This completes the proof.

APPENDIX G

PROOF OF COROLLARY 2

According to Table II, it is obvious that $s.d.o.f. = 0$ can only happen in the last case, where

$$\begin{aligned} N_a &\leq N_b + N_e - N_j, N_b < N_j \leq N_e + N_b - N_a \\ \text{or} \quad N_a &\leq N_e, N_j \leq N_b. \end{aligned} \quad (60)$$

Thus, to complete the proof, we only need to focus on the case of (60), where $s.d.o.f. = \min\{s, \lfloor \frac{N_b}{2} \rfloor\}$ in which $s = \min\{N_a, N_e\} + \min\{N_j, N_e\} - \min\{N_a + N_j, N_e\}$. Note that the case $N_b + N_e - N_j < N_a \leq N_e, N_b < N_j \leq N_e + N_b - N_a$ never happens, so (60) is equivalent to

$$N_a \leq N_e, N_j \leq N_e + N_b - N_a, \quad (61)$$

which implies $s = N_e + \min\{N_j, N_e\} - \min\{N_a + N_j, N_e\}$. Therefore,

- 1) for the case of $N_e \geq N_a + N_j$, $s = 0$ thus $s.d.o.f. = 0$;
- 2) for the case of $N_e \leq N_j$, $s = N_a$ thus $s.d.o.f. = \min\{N_a, \lfloor \frac{N_b}{2} \rfloor\}$;
- 3) for the case of $N_j < N_e < N_a + N_j$, $s = N_a + N_j - N_e$ thus $s.d.o.f. = \min\{N_a + N_j - N_e, \lfloor \frac{N_b}{2} \rfloor\}$.

In conclusion, when $N_b = 1$, $s.d.o.f. = 0$ if and only if $N_e \geq N_a + N_j - 1$; when $N_b > 1$, $s.d.o.f. = 0$ if and only if $N_e \geq N_a + N_j$. This completes the proof.

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